CHAPTER

5

DISCRETE RANDOM VARIABLES

Syllabus coverage

TOPIC 3.3: DISCRETE RANDOM VARIABLES

General discrete random variables

3.3.1 develop the concepts of a discrete random variable and its associated probability function, and their use in modelling data
3.3.2 use relative frequencies obtained from data to obtain point estimates of probabilities associated with a discrete random variable
3.3.3 identify uniform discrete random variables and use them to model random phenomena with equally likely outcomes
3.3.4 examine simple examples of non-uniform discrete random variables
3.3.5 identify the mean or expected value of a discrete random variable as a measurement of centre, and evaluate it in simple cases
3.3.6 identify the variance and standard deviation of a discrete random variable as measures of spread, and evaluate them using technology
3.3.7 examine the effects of linear changes of scale and origin on the mean and the standard deviation
3.3.8 use discrete random variables and associated probabilities to solve practical problems

Bernoulli distributions

3.3.9 use a Bernoulli random variable as a model for two-outcome situations
3.3.10 identify contexts suitable for modelling by Bernoulli random variables
3.3.11 determine the mean p and variance $p(1 - p)$ of the Bernoulli distribution with parameter p
3.3.12 use Bernoulli random variables and associated probabilities to model data and solve practical problems

Binomial distributions

3.3.13 examine the concept of Bernoulli trials and the concept of a binomial random variable as the number of 'successes' in n independent Bernoulli trials, with the same probability of success p in each trial
3.3.14 identify contexts suitable for modelling by binomial random variables
3.3.15 determine and use the probabilities $P(X = x) = \binom{n}{x} p^x(1 - p)^{n-x}$ associated with the binomial distribution with parameters n and p; note the mean np and variance $np(1 - p)$ of a binomial distribution
3.3.16 use binomial distributions and associated probabilities to solve practical problems

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Video playlists (5):

- **5.1** Review of probability
- **5.2** Discrete probability distributions
- **5.3** Measures of centre and spread
- **5.4** The Bernoulli and binomial distributions

WACE question analysis Discrete random variables

Worksheets (13):

- **5.2** Random variables • Discrete probability distributions 1 • Discrete probability distributions 2
- **5.3** Expected values • Expected values using a calculator • Using expected values • Expected value • Variance and standard deviation
- **5.4** The Bernoulli distribution • Binomial probability experiments • Using the binomial probability distribution • Binomial probability – Mean and standard deviation • Applications of binomial probability

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

Video playlist
Review of probability

5.1 Review of probability

The probability of an event occurring must have a value between 0 and 1, where 0 represents an impossible event and 1 represents a certain event.

Probability of an event

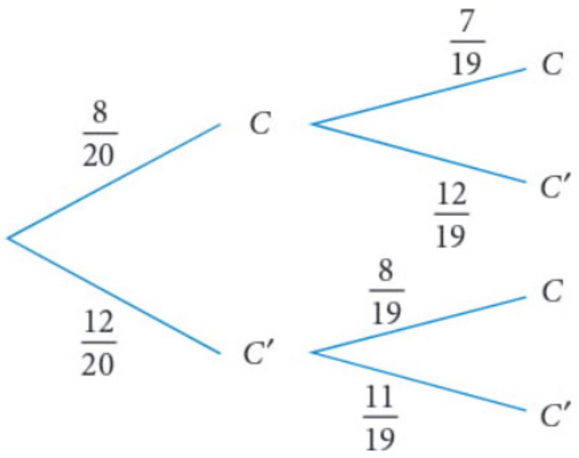
$$\mathrm{P}(A) = \frac{\text{number of ways } A \text{ can occur}}{\text{total number of possible outcomes}}$$

Tree diagrams for compound events

A **compound event** consists of two or more simple events being considered together, such as tossing 2 tails on 3 coins, or winning first or second prize in a raffle. A **tree diagram** is a useful way of representing compound events.

WORKED EXAMPLE 1 Finding probabilities using a tree diagram

A barrel contains 20 balls, of which 8 are multicoloured. Two balls are randomly selected from the barrel, **without replacement**. This means that a ball is selected and the ball is **not** replaced **before** the next ball is selected. Find the probability that one of the two balls is multicoloured.

Steps	**Working**
1 On each selection, the ball selected can be either multicoloured (C) or *not multicoloured* (C'). The first selection is made from 8 balls that are multicoloured and 12 balls that are not multicoloured. In the second selection, the number of multicoloured and not multicoloured balls is determined by the type of ball selected first.	On the first selection: $n(C) = 8, n(C') = 12$ $\mathrm{P}(C) = \frac{8}{20}, \mathrm{P}(C') = \frac{12}{20}$ On the second selection: If the first ball selected is multicoloured: $n(C) = 7, n(C') = 12$ $\mathrm{P}(C) = \frac{7}{19}, \mathrm{P}(C') = \frac{12}{19}$ If the first ball selected is not multicoloured: $n(C) = 8, n(C') = 11$ $\mathrm{P}(C) = \frac{8}{19}, \mathrm{P}(C') = \frac{11}{19}$
2 Represent with a tree diagram.	
3 Identify the branches where there is one multicoloured ball and one ball that is not multicoloured.	$\mathrm{P}(1 \text{ multicoloured}) = \mathrm{P}(CC') + \mathrm{P}(C'C)$
4 Multiply the probabilities along the branches and add the products.	$\mathrm{P}(1 \text{ multicoloured}) = \frac{8}{20} \times \frac{12}{19} + \frac{12}{20} \times \frac{8}{19}$ $= \frac{48}{95}$

CAS can be used to calculate the probabilities when selections are done **without replacement**. This method uses combinations ${}^nC_r$ or $\binom{n}{r}$, which gives the number of ways of choosing r objects from n objects.

USING CAS 1 Selection without replacement probabilities

A bag contains 8 black discs and 7 white discs. If 3 discs are chosen from the bag, without replacement, find the probability of choosing exactly 2 black discs.

Two black discs must be chosen from 8 black discs in the bag and one white disc must be chosen from 7 white discs in the bag.

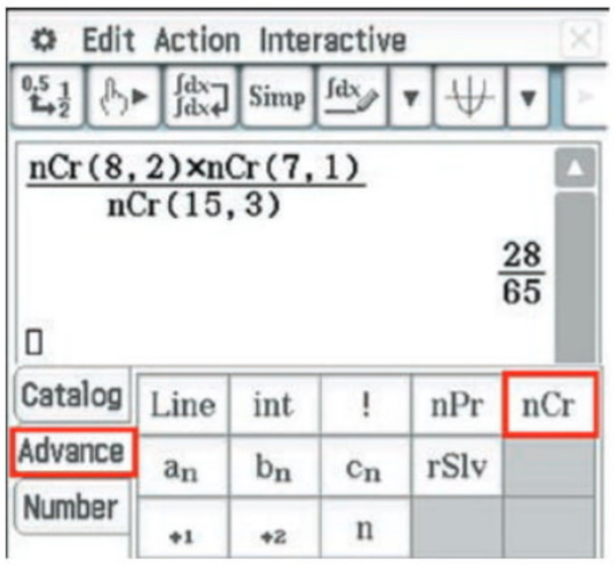
Combinations can be used to find the number of ways this can occur.

$$n(2\text{B}, 1\text{W}) = \binom{8}{2} \times \binom{7}{1}$$

The total number of ways of choosing 3 discs from 15 discs is $\binom{15}{3}$.

$$\text{P}(2\text{B}, 1\text{W}) = \frac{\binom{8}{2} \times \binom{7}{1}}{\binom{15}{3}}$$

ClassPad



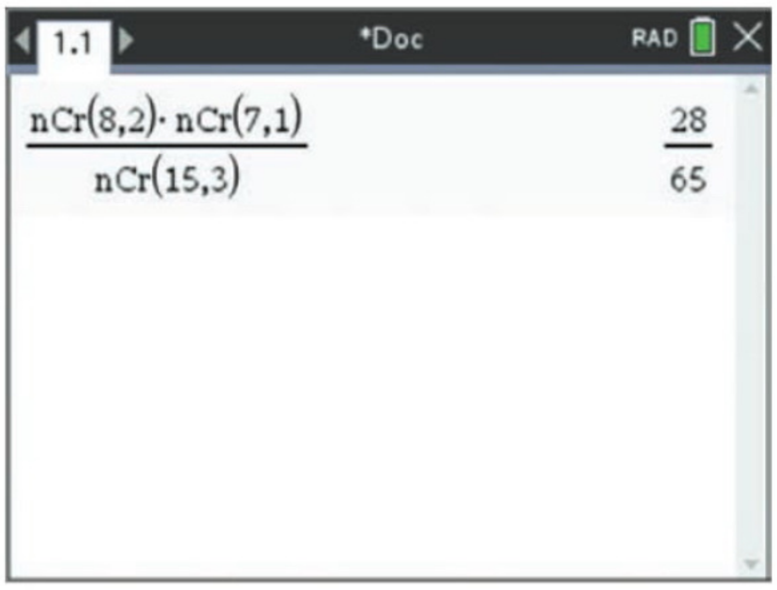
$$\frac{\text{nCr}(8,2)\times\text{nCr}(7,1)}{\text{nCr}(15,3)} \qquad \frac{28}{65}$$

1 Open the soft Keyboard and tap the downward arrow.

2 Tap **Advance** and find the **nCr** symbol.

3 Enter the combinations and values as shown above.

TI-Nspire



$$\frac{\text{nCr}(8,2)\cdot\text{nCr}(7,1)}{\text{nCr}(15,3)} \qquad \frac{28}{65}$$

1 Press **menu** > **Probability** > **Combinations**.

2 Enter the combinations and values as shown above.

The probability is $\frac{28}{65}$.

Arrays

An **array**, **grid** or lattice is a set of numbers arranged in row and column format. It can be useful to show the outcomes of a two-step experiment. An array is a better choice than a tree diagram to represent situations where two simple events would result in a large number of branches.

WORKED EXAMPLE 2 Finding probabilities using an array

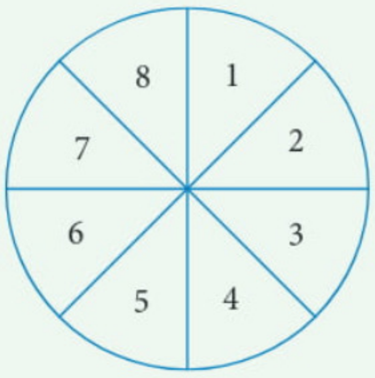
The spinner shown is spun twice.

a Illustrate the sample space using an array.

b Use this diagram to find the probability that

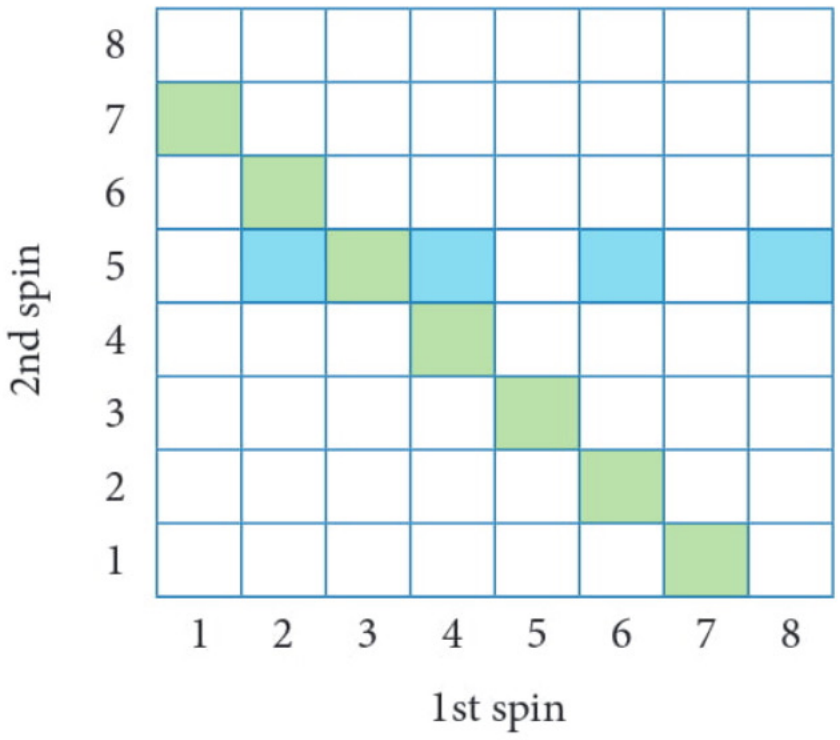
  i the first spin is even and the second spin is a 5

  ii the sum of the two numbers is 8.



Steps | **Working**

a Draw an 8 × 8 grid and let the columns represent the first spin and the rows represent the second spin.



b **i** Identify the four cells in the array where the first number is even and the second number is a 5. Use this to calculate the probability. These cells are shaded blue in the grid.

$$P(\text{even}, 5) = \frac{4}{64} = \frac{1}{16}$$

ii Identify the seven cells in the array where the first and second numbers sum to 8. The cells in which the sum is 8 are shaded green.

$$P(\text{sum of } 8) = \frac{7}{64}$$

Independent events

Two events A and B are **independent** if one event does not influence the outcome of the other event. For example, where event A is Fremantle Dockers winning a football match at home and event B is West Coast Eagles winning a different football match in Adelaide.

If $P(A) = 0.6$ and $P(B) = 0.5$ then the probability Fremantle Dockers AND West Coast Eagles both win can be found using the multiplication rule.

$P(A \cap B) = 0.6 \times 0.5 = 0.3$.

Independent events

For two **independent events** A and B, $P(A \cap B) = P(A) \times P(B)$.

WORKED EXAMPLE 3 **Finding probabilities for independent events**

Jordyn is training to improve the accuracy of her tennis serve by attempting to hit a can on the centre line. The probability of hitting the can on any attempt is 0.6 and her success on any serve is independent of the result on the previous serve. Find the probability Jordyn hits the can on two out of three attempts.

Steps	Working
1 Let event H represent Jordyn hits the can and event M represent Jordyn misses the can. List the different ways of getting two hits and one miss.	two hits = $\{H, H, M\}$ or $\{H, M, H\}$ or $\{M, H, H\}$
2 The three serves are independent so multiply the probabilities for each combination of three shots.	$\text{P}(H, H, M) = 0.6 \times 0.6 \times 0.4 = 0.144$ $\text{P}(H, M, H) = 0.6 \times 0.4 \times 0.6 = 0.144$ $\text{P}(M, H, H) = 0.4 \times 0.6 \times 0.6 = 0.144$
3 Add the probabilities for each possibility where there are two hits.	P(2 hits and 1 miss) $= 0.144 + 0.144 + 0.144 = 0.432$

The addition rule for probability

$\text{P}(A \cup B)$ means the probability of events A or B or both occurring.

The addition rule for probability

The **addition rule for probability** is

$$\text{P}(A \cup B) = \text{P}(A) + \text{P}(B) - \text{P}(A \cap B)$$

WORKED EXAMPLE 4 **Finding probabilities using the addition rule**

If $\text{P}(A \cup B) = \frac{2}{5}$, $\text{P}(A \cap B) = \frac{1}{5}$ and $\text{P}(A) = 2 \times \text{P}(B)$, find $\text{P}(A)$.

Steps	Working
1 Let $\text{P}(B) = b$. Write $\text{P}(A)$ in terms of b.	Let $\text{P}(B) = b$. $\text{P}(A) = 2 \times \text{P}(B) = 2b$
2 Substitute into the addition rule.	$\text{P}(A \cup B) = \text{P}(A) + \text{P}(B) - \text{P}(A \cap B)$ $\frac{2}{5} = b + 2b - \frac{1}{5}$ $\frac{3}{5} = 3b$ $b = \frac{1}{5}$
3 Substitute the value of b into $\text{P}(A) = 2b$.	$\text{P}(A) = 2 \times \frac{1}{5} = \frac{2}{5}$

Conditional probability

The probability of an event occurring if another event has also occurred is a **conditional probability**. In these situations, the sample space is reduced because of the condition placed on the probability.

Conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

where $P(A|B)$ means 'the probability of A occurring, given B occurs'.

If events A and B are independent, then $P(A|B) = P(A)$.

WORKED EXAMPLE 5 **Finding conditional probabilities**

Farmer Bob owns a tractor, a motorcycle and a utility. Each morning he starts each vehicle but they do not always start on the first attempt. The respective probabilities of each starting on the first attempt are 0.2, 0.7 and 0.6. On Wednesday morning, two vehicles started on the first attempt. Find the probability the two that started were the tractor and the utility.

Steps	Working
1 Write the conditional probability in the form $P(A\|B) = \frac{P(A \cap B)}{P(B)}$.	$P(\text{tractor and utility}\|\text{two starts})$ $= \frac{P(\text{tractor and utility start AND two starts})}{P(\text{two starts})}$
2 Find the probability that the tractor and the utility start but the motorcycle does not start. Let T = tractor starts M = motorcycle starts and U = utility starts.	$P(T) = 0.2$, $P(M) = 0.7$, $P(U) = 0.6$ $P(M') = 1 - 0.7$ $= 0.3$ $P(T, M', U) = 0.2 \times 0.3 \times 0.6$ $= 0.036$
3 Find the probability that two vehicles start, that is, $P(T, M, U') + P(T, M', U) + P(T', M, U)$.	$P(\text{two starts})$ $= P(T, M, U') + P(T, M', U) + P(T', M, U)$ $= 0.2 \times 0.7 \times 0.4 + 0.036 + 0.8 \times 0.7 \times 0.6$ $= 0.428$
4 Substitute into the conditional probability formula.	$P(\text{tractor and utility}\|\text{two starts}) = \frac{0.036}{0.428}$ $= \frac{9}{107}$

EXERCISE 5.1 Review of probability

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Mastery

1 WORKED EXAMPLE 1 A bag contains 18 tulip bulbs and 12 daffodil bulbs. Three bulbs are selected *without replacement*. Find the probability of selecting three bulbs of the same type.

2 Using CAS 1 A bag contains fifteen red marbles and five green marbles. If four marbles are chosen from the bag, *without replacement*, find the probability of choosing three green marbles.

3 WORKED EXAMPLE 2 A tetrahedral die, with four faces numbered 2, 4, 6 and 8, is rolled twice and the number facedown is noted.

- **a** Illustrate the sample space using an array.
- **b** Use this diagram to find the probability that
  - **i** the same number occurs on both rolls
  - **ii** the sum of the two numbers is 10.

4 WORKED EXAMPLE 3 Emily exercises each morning before school. She will either go for a run or a swim. The probability that she will go for a run on any morning is 0.3 and the type of exercise she does on any morning is independent of the exercise done on the previous morning. Find the probability that Emily will go for a run on either Monday or Tuesday.

5 WORKED EXAMPLE 4 If $P(A \cup B) = 0.7$, $P(A \cap B) = 0.05$ and $P(B) = 4 \times P(A)$, find $P(A)$.

6 WORKED EXAMPLE 5 Four identical balls are numbered 1, 2, 3 and 4 and put into a box. A ball is randomly drawn from the box, and not returned to the box. A second ball is then randomly drawn from the box. Find the probability that the first number drawn is numbered 2 if it is known that the sum of the numbers is at least 4.

Calculator-free

7 (4 marks) Two boxes each contain four stones that differ only in colour. Box 1 contains four black stones and Box 2 contains two black stones and two white stones. A box is chosen randomly, and one stone is drawn randomly from it. Each box is equally likely to be chosen, as is each stone.

a What is the probability that the randomly drawn stone is black? (2 marks)

b It is not known from which box the stone has been drawn. Given that the stone that is drawn is black, what is the probability that it was drawn from Box 1? (2 marks)

8 (3 marks) The only possible outcomes when a coin is tossed are a head or a tail. When an unbiased coin is tossed, the probability of tossing a head is the same as the probability of tossing a tail. Jo has three coins in her pocket; two are unbiased and one is biased. When the biased coin is tossed, the probability of tossing a head is $\frac{1}{3}$. Jo randomly selects a coin from her pocket and tosses it.

a Find the probability that she tosses a head. (2 marks)

b Find the probability that she selected an unbiased coin, given that she tossed a head. (1 mark)

9 (6 marks) Sally aims to walk her dog, Mack, most mornings. If the weather is pleasant, the probability that she will walk Mack is $\frac{3}{4}$ and if the weather is unpleasant, the probability that she will walk Mack is $\frac{1}{3}$.

Assume that pleasant weather on any morning is independent of pleasant weather on any other morning. In a particular week, the weather was pleasant on Monday morning and unpleasant on Tuesday morning.

a Find the probability that Sally walked Mack on at least one of these two mornings. (2 marks)

b **i** In the month of April, the probability of pleasant weather in the morning was $\frac{5}{8}$. Find the probability that on a particular morning in April, Sally walked Mack. (2 marks)

ii Using your answer from part **b i**, or otherwise, find the probability that on a particular morning in April, the weather was pleasant, given that Sally walked Mack that morning. (2 marks)

10 (3 marks) A company produces motors for refrigerators. There are two assembly lines, Line A and Line B. 5% of the motors assembled on Line A are faulty and 8% of the motors assembled on Line B are faulty. In one hour, 40 motors are produced from Line A and 50 motors are produced from Line B. At the end of an hour, one motor is selected at random from all the motors that have been produced during that hour.

a What is the probability that the selected motor is faulty? Express your answer in the form $\frac{1}{b}$, where b is a positive integer. (2 marks)

b The selected motor is found to be faulty. What is the probability that it was assembled on Line A? Express your answer in the form $\frac{1}{c}$, where c is a positive integer. (1 mark)

11 (3 marks) An online shopping site sells boxes of doughnuts.

A box contains 20 doughnuts. There are only four types of doughnuts in the box. They are:

- glazed, with custard
- glazed, with no custard
- not glazed, with custard
- not glazed, with no custard.

It is known that, in the box:

- $\frac{1}{2}$ of the dougnuts are with custard
- $\frac{7}{10}$ of the doughnuts are not glazed
- $\frac{1}{10}$ of the doughnuts are glazed, with custard.

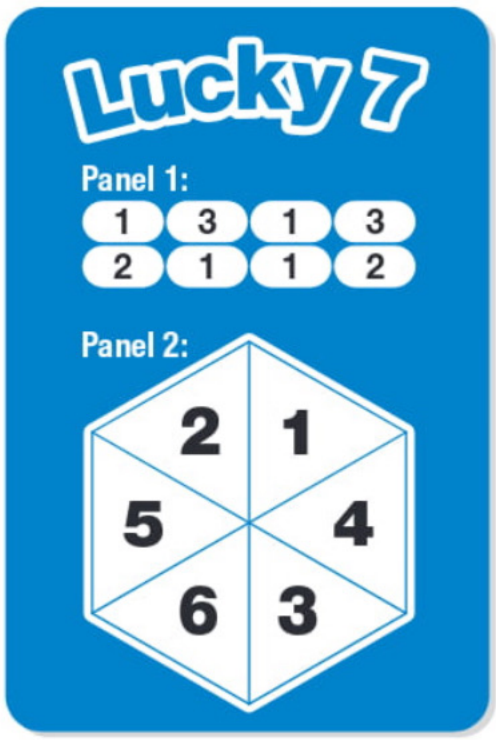
a A doughnut is chosen at random from the box. Find the probability that it is not glazed, with custard. (1 mark)

b The 20 doughnuts in the box are randomly allocated to two new boxes, Box *A* and Box *B*. Each new box contains 10 doughnuts. One of the two new boxes is chosen at random and then a doughnut from that box is chosen at random.

Let g be the number of glazed doughnuts in Box *A*. Find the probability, in terms of g, that the doughnut comes from Box *B* given that it is glazed. (2 marks)

Calculator-assumed

12 © SCSA MM2021 Q10b (2 marks) A charity organisation has printed 'Lucky 7' scratchie tickets as a fundraiser for use at two special events. The tickets contain two panels. Each ticket has the same numbers as the sample ticket shown below, arranged randomly and hidden within each panel.



A player scratches one section of each panel to reveal a number. The two numbers revealed are then added together. If the total is 7 or higher, the player wins a prize.

At the first event, 400 tickets are purchased, and a prize is won on 124 occasions.
Let p denote the probability that a prize is won. Show that the probability p of winning a prize is $\frac{7}{24}$.

13 (3 marks) A box contains five red marbles and three yellow marbles. Two marbles are drawn at random from the box without replacement. Find the probability that the marbles are

- **a** different colours (2 marks)
- **b** the same colour. (1 mark)

14 (3 marks) Demelza is a badminton player. If she wins a game, the probability that she will win the next game is 0.7. If she loses a game, the probability that she will lose the next game is 0.6. Demelza has just won a game.

Find the probability that

- **a** she wins her next two games (1 mark)
- **b** she wins exactly one of her next two games. (2 marks)

5.2 Discrete probability distributions

Video playlist
Discrete probability distributions

Worksheets
Random variables

Discrete probability distributions 1

Discrete probability distributions 2

A **random variable,** X, is a variable whose value is determined by the outcome of a random experiment. There are two types of random variables: **discrete** and **continuous**.

A **discrete random variable** can only take a countable number of values, for example, the number of pets in a household.

A **continuous random variable** can take any value in a given interval, for example, the height of students in a class.

If a coin is tossed three times and the number of heads is recorded, then the discrete random variable X represents the number of heads in three tosses. The values that this variable can take is represented by x, where $x = 0, 1, 2, 3$.

The **probability distribution** of a discrete random variable shows all the possible values of the variable and the probabilities associated with those values. This function can be represented numerically as a table, in graphical form or as a formula.

Let $p(x)$ or $\mathrm{P}(X = x)$ be the probability of obtaining x heads in three tosses of a coin.

The probability distribution of X is

x	0	1	2	3
$\mathrm{P}(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Properties of a probability distribution

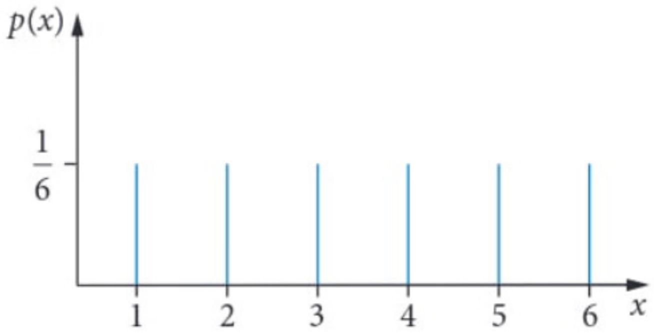
- All probabilities must be between 0 and 1 inclusive: $0 \le p(x) \le 1$.
- The sum of all probabilities must equal 1: $\Sigma p(x) = 1$.

The probability distribution for a uniform discrete random variable

A discrete uniform distribution is one in which all the outcomes have the same probability of occurring. A simple example of a uniform distribution is rolling a die. The discrete random variable X represents the number rolled.

The probability distribution of X is

x	1	2	3	4	5	6
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$



If X is a uniform discrete random variable where $x = 1, 2, 3, \ldots N$ then $P(X = x) = \frac{1}{N}$.

WORKED EXAMPLE 6 **Finding the probability distribution of a uniform discrete random variable**

A disc is randomly selected from a bag that contains five discs numbered 1, 2, 3, 4 and 5. The random variable X represents the number selected.

a Identify the probability distribution of X.

b Complete the probability distribution of X.

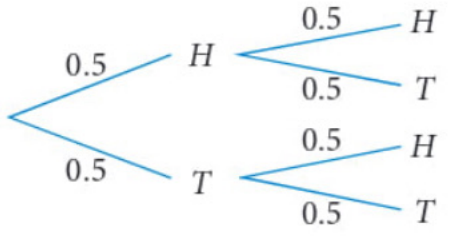
x						
$P(X = x)$						

Steps	Working
a A discrete uniform distribution is one in which all the outcomes have the same probability of occurring.	This is a discrete uniform distribution.
b There are five outcomes and each has the same probability of $\frac{1}{5}$ of occurring.	

x	1	2	3	4	5
$P(X = x)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

Non-uniform distributions

Many discrete random variables do not have uniform distributions. Consider the situation where we toss a coin twice and the discrete random variable X represents the number of heads that occur.



x	0	1	2
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

The probability of rolling one head is not the same as the probability of rolling no heads or the probability of rolling two heads. This discrete random variable does not have a uniform distribution.

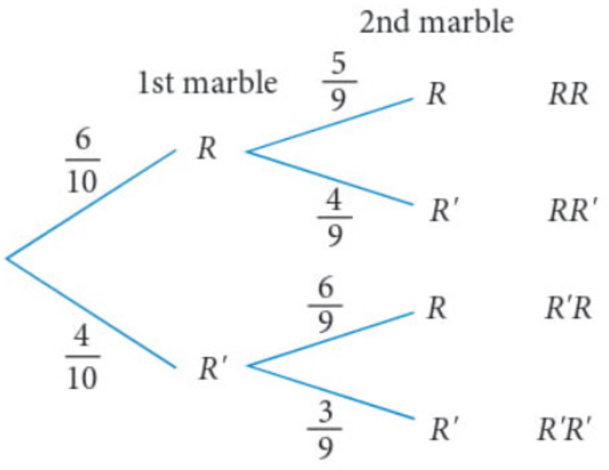
WORKED EXAMPLE 7 Finding the probability distribution of a non-uniform discrete random variable

A bag contains ten marbles of which six are red. Two marbles are taken from the bag. If X represents the number of red marbles taken, list the probability distribution of X.

x	0	1	2
$P(X = x)$			

Steps | **Working**

1 Draw a tree diagram to illustrate the problem.



2 Use the tree diagram to find the probabilities.

$$P(X = 0) = \frac{4}{10} \times \frac{3}{9} = \frac{2}{15}$$

$$P(X = 1) = \frac{6}{10} \times \frac{4}{9} + \frac{4}{10} \times \frac{6}{9} = \frac{8}{15}$$

$$P(X = 2) = \frac{6}{10} \times \frac{5}{9} = \frac{1}{3}$$

3 Write the probabilities in the table.

The probability distribution of X is

x	0	1	2
$P(X = x)$	$\frac{2}{15}$	$\frac{8}{15}$	$\frac{1}{3}$

The probabilities could also be calculated using combinations:

$$P(X = 1) = \frac{\binom{4}{1} \times \binom{6}{1}}{\binom{10}{2}} = \frac{8}{15}$$

Exam hack

Always check that the sum of all probabilities is 1.

WORKED EXAMPLE 8 Finding a probability for a given probability distribution

The probability distribution of a discrete random variable, X, is given by the table below.

Find the value of p.

x	1	2	3	4
$P(X = x)$	$p^2 - 0.1$	p	$p^2 - 0.1$	0.2

Steps	Working
1 As this is a probability distribution, the sum of the probabilities is one.	$\Sigma p(x) = 1$ $p^2 - 0.1 + p + p^2 - 0.1 + 0.2 = 1$ $2p^2 + p - 1 = 0$
2 Factorise and solve for p.	$(2p - 1)(p + 1) = 0$ $p = -1 \text{ or } \frac{1}{2}$
3 As p is a probability, $p \geq 0$.	$p = \frac{1}{2}$

WORKED EXAMPLE 9 Compound event probabilities

The Mercury newspaper publishes three puzzles every day. John attempts these puzzles each day and has calculated the probability of solving 0, 1, 2 or 3 puzzles. The probability distribution of the number of puzzles solved X, is given in the table below.

Find the probability that John solves the same number of puzzles on Monday and Tuesday.

x	0	1	2	3
$P(X = x)$	0.4	0.3	0.2	0.1

Steps	Working
1 List the possible options where John solves the same number of puzzles on two days. P(0, 0) is the probability of 0 puzzles solved on Monday and on Tuesday.	P(same on Mon and Tues) $= P(0,0) + P(1,1) + P(2,2) + P(3,3)$
2 Find the probabilities from the table and multiply the probabilities for each pair.	$= 0.4 \times 0.4 + 0.3 \times 0.3 + 0.2 \times 0.2 + 0.1 \times 0.1$ $= 0.16 + 0.09 + 0.04 + 0.01$ $= 0.3$

EXERCISE 5.2 Discrete probability distributions

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Recap

1 A box contains four red marbles and two yellow marbles. Two marbles are drawn at random from the box without replacement. The probability that the marbles are the same colour is

A $\frac{7}{30}$ **B** $\frac{2}{5}$ **C** $\frac{7}{15}$ **D** $\frac{8}{15}$ **E** $\frac{5}{9}$

2 Two cubes are randomly selected, with replacement, from a bag that contains seven cubes numbered 1, 2, 3, 4, 5, 6 and 7. The discrete random variable X is the sum of the two selected cubes. Find

a $P(X = 5)$ **b** $P(X \geq 5)$

Mastery

3 WORKED EXAMPLE 6 A disc is randomly selected from a bag that contains eight discs numbered 1, 2, 3, 4, 5, 6, 7 and 8. The random variable X represents the number selected.

a Identify the probability distribution of X.

b List the probability distribution of X.

4 WORKED EXAMPLE 7 A coin is biased so that the probability of tossing a tail is $\frac{2}{5}$. List the probability distribution of the number of tails that are obtained on two tosses of the coin.

5 Julie has a bag that contains three blue discs and five yellow discs. She selects two discs, without replacement. The discrete random variable, X, represents the number of yellow discs selected. List the probability distribution of X.

6 WORKED EXAMPLE 8 The probability distribution of a discrete random variable, X, is given by the table below.

x	0	1	2	3
$P(X = x)$	p	$0.3p$	$p^2 - 0.2$	$p^2 - 0.3$

Find the value of p.

7 The probability distribution function of a discrete random variable, X, is given by the function

$f(x) = kx, x = 1, 2, 3, 4.$

Find the value of k.

8 WORKED EXAMPLE 9 Emma drives through two intersections with traffic lights on her way to work. The probability distribution of the number of green lights that Emma gets on any trip is given below.

x	0	1	2
$P(X = x)$	0.5	0.3	0.2

Find the probability that on two successive trips Emma gets a total of two green lights.

Calculator-free

9 (3 marks) The probability distribution of a discrete random variable, X, is given by the table below.

x	0	1	2	3	4
$P(X = x)$	0.2	$0.6p^2$	0.1	$1 - p$	0.1

Show that $p = \dfrac{2}{3}$ or $p = 1$.

10 (3 marks) The discrete random variable, X, has the probability distribution given by the table below.

x	−1	0	1	2
$P(X = x)$	p^2	p^2	$\dfrac{p}{4}$	$\dfrac{4p+1}{8}$

Find the value of p.

11 (2 marks) Jane drives to work each morning and passes through three intersections with traffic lights. The number, X, of traffic lights that are red when Jane is driving to work is a random variable with probability distribution given by

x	0	1	2	3
$P(X = x)$	0.1	0.2	0.3	0.4

Jane drives to work on two consecutive days. What is the probability that the number of traffic lights that are red is the same on both days?

12 (4 marks) On any given day, the number, X, of telephone calls that Daniel receives is a random variable with probability distribution given by

x	0	1	2	3
$P(X = x)$	0.2	0.2	0.5	0.1

a What is the probability that Daniel receives only one telephone call on each of three consecutive days? (1 mark)

b Daniel receives telephone calls on both Monday and Tuesday. What is the probability that Daniel receives a total of four calls over these two days? (3 marks)

Calculator-assumed

13 © SCSA MM2016 Q15abii (4 marks) A tetrahedral die has the numbers 1 to 4 on each face. When thrown, each side is equally likely to land facedown. Let X be defined as the sum of the numbers on the facedown side when the die is thrown twice.

a Copy and complete the following table. (1 mark)

	Roll two				
	Sum of two rolls	1	2	3	4
Roll one	1	$1 + 1 = 2$	3		
	2	3			
	3		5		
	4				

b **i** Hence, or otherwise, complete the probability distribution of X, which is given by the following table. (1 mark)

x	2	3	4	5	6	7	8
$P(X = x)$	$\frac{1}{16}$						$\frac{1}{16}$

ii Calculate the probability of obtaining a sum of five or less. (2 marks)

5.3 Measures of centre and spread

The three **measures of centre** of a probability distribution are the **mean**, the **mode** and the **median**. They measure the expected or likely outcome for the distribution. The mean is the only measure of centre covered in this course.

The expected value (mean)

The mean of a discrete random variable is called the **expected value**. For a discrete random variable, X, the expected value is written in mathematical notation as $E(X)$ or μ.

The expected value is found by summing the product of x and $P(X = x)$ for all possible values of x.

Remember that $p(x)$ is a short way of writing $P(X = x)$.

The expected value of a discrete probability distribution

$$\mu = E(X) = \Sigma x \cdot p(x)$$

Video playlist
Measures of centre and spread

Worksheets
Expected values

Expected values using a calculator

Using expected values

Expected value

The effects of linear changes of scale and origin on the expected value of $aX + b$

The discrete random variable X represents the number selected at random from the set $\{1, 2, 3\}$. The probability distribution of X is

x	1	2	3
$p(x)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

The discrete random variable X has a mean $\mu = E(X) = 2$.

Change the scale

If we change the scale and multiply all these values by 2, then the distribution of $2X$ is

$2x$	2	4	6
$p(x)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

The discrete random variable $2X$ has a mean $\mu = E(2X) = 2 \times 2 = 4$.

Change the origin

If we now add 5 to all the $2x$ values then the distribution of $2X + 5$ is

$2x + 5$	7	9	11
$p(x)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

The discrete random variable $2X + 5$ has a mean $\mu = E(2X + 5) = 2 \times 2 + 5 = 9$.

Expected value of $aX + b$

For a linear function $aX + b$ of a discrete random variable X:

$$E(aX + b) = aE(X) + b$$

WORKED EXAMPLE 10 Finding the expected value from a discrete probability distribution

The probability distribution of a discrete random variable X is shown below.

x	0	1	2	3
$P(X = x)$	0.25	0.3	0.2	0.25

Find

a the mean of X

b $E(10X + 20)$

c $P(X \leq \mu)$.

Steps

a **1** Rewrite the table with rows as columns and add a column headed $x \times p(x)$. Calculate the product of the x values and their probabilities.

2 The sum of the $x \times p(x)$ column is the expected value of X or $E(X)$.

Working

x	$p(x)$	$x \times p(x)$
0	0.25	0
1	0.3	0.3
2	0.2	0.4
3	0.25	0.75
Total		1.45

$\mu = E(X) = \Sigma x \cdot p(x)$

The mean or $E(X)$ is 1.45.

b Substitute into the formula:

$E(aX + b) = aE(X) + b$

$$\begin{aligned} E(10X + 20) &= 10E(X) + 20 \\ &= 10 \times 1.45 + 20 \\ &= 34.5 \end{aligned}$$

c Identify the x values in the table that are less than or equal to 1.45 and add their probabilities.

$$\begin{aligned} P(X \leq \mu) &= P(X \leq 1) \\ &= P(X = 0) + P(X = 1) \\ &= 0.25 + 0.3 \\ &= 0.55 \end{aligned}$$

WORKED EXAMPLE 11 Finding probabilities given the expected value of a discrete probability distribution

The probability distribution of a discrete random variable X is shown below.

x	1	2	3	4
$P(X = x)$	a	0.1	0.2	b

The mean of the distribution is 2.4. Find the values of a and b.

Steps

1 Rewrite the table with rows as columns and add a column headed $x \times p(x)$.

Working

x	$p(x)$	$x \times p(x)$
1	a	a
2	0.1	0.2
3	0.2	0.6
4	b	$4b$
Total		

2 Calculate the product of the x values and their probabilities.

The sum of the $p(x)$ column must be 1.

The sum of the $x \times p(x)$ column must be 2.4.

x	$p(x)$	$x \times p(x)$
1	a	a
2	0.1	0.2
3	0.2	0.6
4	b	$4b$
Total	1.0	2.4

The mean E(X) is 2.4.

3 Write the equations for the total of the $p(x)$ and $x \times p(x)$ columns.

$$a + 0.1 + 0.2 + b = 1$$

$$a + b = 0.7 \qquad [1]$$

$$a + 0.2 + 0.6 + 4b = 2.4$$

$$a + 4b = 1.6 \qquad [2]$$

4 Solve the simultaneous equations [1] and [2] to find the values of a and b.

[2] − [1] gives

$$3b = 0.9$$

$$b = 0.3$$

Substitute in [1]:

$$a + 0.3 = 0.7$$

$$a = 0.4$$

So, $a = 0.4$, $b = 0.3$.

The variance and standard deviation

Worksheet
Variance and standard deviation

The most useful measures of spread for a probability distribution are the **variance** and the **standard deviation**.

The variance and standard deviation of a discrete random variable, X, measure the spread of the variable about its mean.

$$\text{Var}(X) = \sigma^2 = \Sigma(x - \mu)^2 p(x)$$

$$\text{SD}(X) = \sqrt{\text{Var}(X)}$$

There is another formula, called the computational formula, that can be used to find the variance. The computational formula for variance can be found algebraically.

$$\begin{aligned}\sum(x - \mu)^2 p(x) &= \text{E}((X - \mu)^2)\\ &= \text{E}(X^2 - 2\mu X + \mu^2)\end{aligned}$$

Using E($aX + b$) = aE(X) + b, this expression can be simplified to

$$\begin{aligned}&= \text{E}(X^2) - \text{E}(2\mu X) + \text{E}(\mu^2)\\ &= \text{E}(X^2) - 2\mu\text{E}(X) + \mu^2\\ &= \text{E}(X^2) - 2\mu^2 + \mu^2\\ &= \text{E}(X^2) - \mu^2\end{aligned}$$

Another form of the variance formula is $\mathrm{Var}(X) = \mathrm{E}(X^2) - \mu^2$.

The variance formula

A simpler computational formula for the variance is:

$$\mathrm{Var}(X) = \sigma^2 = \mathrm{E}(X^2) - \mu^2$$

where $\mathrm{E}(X^2) = \Sigma x^2 \cdot p(x)$ and $\mu = \mathrm{E}(X)$.

standard deviation $\mathrm{SD}(X) = \sqrt{\mathrm{Var}(X)}$

The symbols for the variance and standard deviation are $\sigma^2 = \mathrm{Var}(X)$ and $\sigma = \mathrm{SD}(X)$.

Standard deviation is the square root of the variance.

Exam hack

The formula sheet has the variance formula given as:

$$\mathrm{Var}(X)\text{: } \sigma^2 = \Sigma(x - \mu)^2 p(x)$$

The computational formula

$$\mathrm{Var}(X)\text{: } \sigma^2 = \mathrm{E}(X^2) - \mu^2$$

is more useful when calculating the variance of a discrete distribution where the mean μ is a fraction or decimal.

WORKED EXAMPLE 12 Finding the variance and standard deviation of a discrete random variable X, using the computational formula: $\mathrm{Var}(X)$: $\sigma^2 = \mathrm{E}(X^2) - \mu^2$

For the probability distribution of a discrete random variable X, find the

x	0	1	2	3	4
$p(x)$	0.4	0.1	0.1	0.2	0.2

a expected value

b variance

c standard deviation, correct to three decimal places.

Steps

Working

a **1** Add two extra columns headed $x \times p(x)$ and $x^2 \times p(x)$ for calculating $\mathrm{E}(X)$ and $\mathrm{E}(X^2)$ respectively.

2 Multiply x by $p(x)$, enter the results in the $x \times p(x)$ column and find the total.

3 Multiply x by $x \times p(x)$, enter the results in the $x^2 \times p(x)$ column and find the total.

4 The total of the $x \times p(x)$ column is $\mathrm{E}(X)$.

x	$p(x)$	$x \times p(x)$	$x^2 \times p(x)$
0	0.4	0	0
1	0.1	0.1	0.1
2	0.1	0.2	0.4
3	0.2	0.6	1.8
4	0.2	0.8	3.2
Total	1.0	$\mathrm{E}(X) = 1.7$	$\mathrm{E}(X^2) = 5.5$

$\mathrm{E}(X) = \Sigma x \cdot p(x) = 1.7$

b **1** The total of the $x^2 \times p(x)$ column is $\mathrm{E}(X^2)$.

$\mathrm{E}(X^2) = 5.5$

2 Use the computational formula to find $\mathrm{Var}(X)$.

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - \mu^2$$
$$\mathrm{Var}(X) = 5.5 - 1.7^2$$
$$= 2.61$$

c Find the standard deviation using the formula:

$\mathrm{SD}(X) = \sqrt{\mathrm{Var}(X)}$

$\mathrm{SD}(X) = \sqrt{2.61} \approx 1.616$

WORKED EXAMPLE 13 Finding the variance and standard deviation of a discrete random variable X, using the formula: Var(X): $\sigma^2 = \Sigma(X - \mu)^2 p(x)$

Find the variance of the discrete random variable X.

x	0	1	2	3
$p(x)$	0.5	0.2	0.1	0.2

Steps

1. Add two extra columns headed $x \times p(x)$ and $(x - \mu)^2 \times p(x)$ for calculating E(X) and Var(X) respectively.
2. Multiply x by $p(x)$, enter the results in the $x \times p(x)$ column and find the total. The mean $\mu = 1$, which makes it easier to use the formula $\sigma^2 = \Sigma(x - \mu)^2 p(x)$.
3. Subtract E(X) from x, square, then multiply the result by $p(x)$, enter the results in the $(x - \mu)^2 \times p(x)$ column and find the total.
4. The total of the $x \times p(x)$ column is the mean and the total of $(x - \mu)^2 \times p(x)$ is the variance.

Working

x	$p(x)$	$x \times p(x)$	$(x - \mu)^2 \times p(x)$
0	0.5	0	$(-1)^2 \times 0.5 = 0.5$
1	0.2	0.2	$(0)^2 \times 0.2 = 0$
2	0.1	0.2	$(1)^2 \times 0.1 = 0.1$
3	0.2	0.6	$(2)^2 \times 0.2 = 0.8$
		$\mu = 1$	$\sigma^2 = 1.4$

Var(X): $\sigma^2 = 1.4$

The effects of linear changes of scale and origin on the variance of $aX + b$

Earlier in this exercise we examined the effects of linear changes of scale and origin on the mean of $aX + b$ for the probability distribution of a discrete random variable X. We will now look at how these changes affect the variance of a distribution.

The probability distribution of the discrete random variable X is

x	1	2	3
$p(x)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

The variance is the average of the squared deviations from the mean.

The discrete random variable X has a variance $\sigma^2 = \frac{2}{3}$.

Change the scale

If we change the scale and multiply all these values by 2, then the distribution of $2X$ is

$2x$	2	4	6
$p(x)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

When we multiply the x values by 2 the deviations from the mean also multiply by 2 and the squared deviations multiply by $2^2 = 4$.

The discrete random variable $2X$ has a variance $\sigma^2 = \frac{2}{3} \times 2^2 = \frac{8}{3}$.

Change the origin

If we now add 5 to all the $2x$ values, then the distribution of $2X + 5$ is

$2x + 5$	7	9	11
$p(x)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

When we add 5 to all the $2x$ values, the spread of the $2x$ values will not change.

The discrete random variable $2X + 5$ also has a variance $\sigma^2 = \frac{2}{3} \times 2^2 = \frac{8}{3}$.

Variance of $aX + b$

For a linear function $aX + b$ of a discrete random variable X:

$$\text{Var}(aX + b) = a^2\,\text{Var}(X)$$

WORKED EXAMPLE 14 Finding the expected value and variance of $aX + b$

The scores X, on a fitness test at school, are a discrete random variable. The variable X has an expected value of 12.5 and a variance of 2.5.

Find

a $E(3X + 4)$

b $\text{Var}(3X + 4)$.

The scores need to be rescaled to be compatible with other classes, so the teacher applies the scaling $Y = aX + b$, where a and b are positive constants, so that the mean is now 80 and the variance 40.

c Find the values of a and b.

Steps	**Working**
a Use the formula: $E(aX + b) = a\,E(X) + b$	$E(3X + 4) = 3\,E(X) + 4$ $= 3 \times 12.5 + 4$ $= 41.5$
b Use the formula: $\text{Var}(aX + b) = a^2\,\text{Var}(X)$	$\text{Var}(3X + 4) = 3^2 \times \text{Var}(X)$ $= 9 \times 2.5$ $= 22.5$
c **1** Substitute $E(aX + b) = 80$ into $E(aX + b) = aE(X) + b$ and $\text{Var}(aX + b) = 40$ into $\text{Var}(aX + b) = a^2\text{Var}(X)$.	$E(aX + b) = aE(X) + b$ $80 = 12.5a + b$ [1] $\text{Var}(aX + b) = a^2\,\text{Var}(X)$ $40 = 2.5a^2$ [2]
2 Solve the simultaneous equations.	From [2]: $a^2 = 16$ $a = 4$ Substitute into [1]: $12.5 \times 4 + b = 80$ $50 + b = 80$ $b = 30$

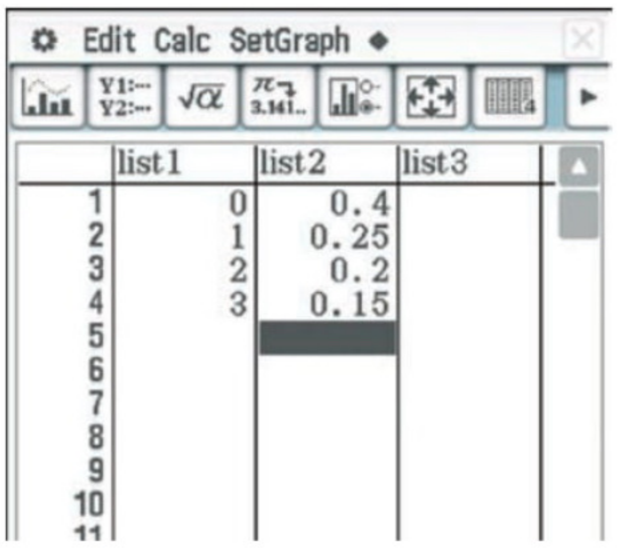
CAS can be used to find the expected value, the variance and the standard deviation of a random variable X, given its probability distribution.

USING CAS 2 Expected value, variance and standard deviation

Using the results from the following table, find the mean, variance and standard deviation of X.

x	0	1	2	3
$P(X = x)$	0.4	0.25	0.2	0.15

ClassPad

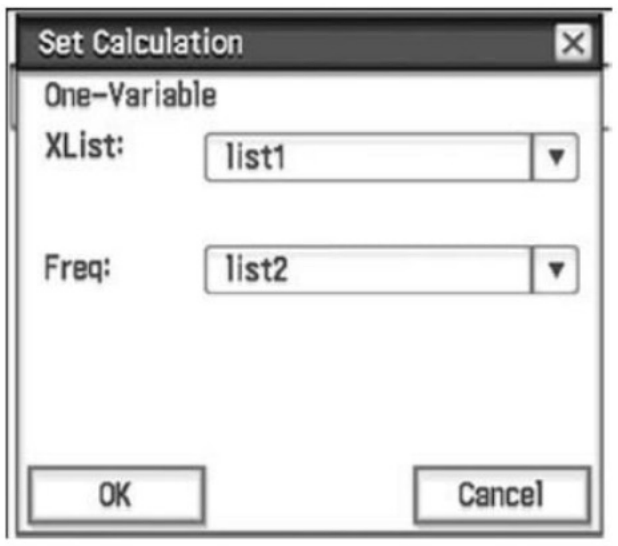


	list1	list2	list3
1	0	0.4	
2	1	0.25	
3	2	0.2	
4	3	0.15	
5			
6			
7			
8			
9			
10			

1 Tap **Menu** > **Statistics**.

2 Clear all lists.

3 Enter the values from the table into **list1** and **list2** as shown above.



One-Variable

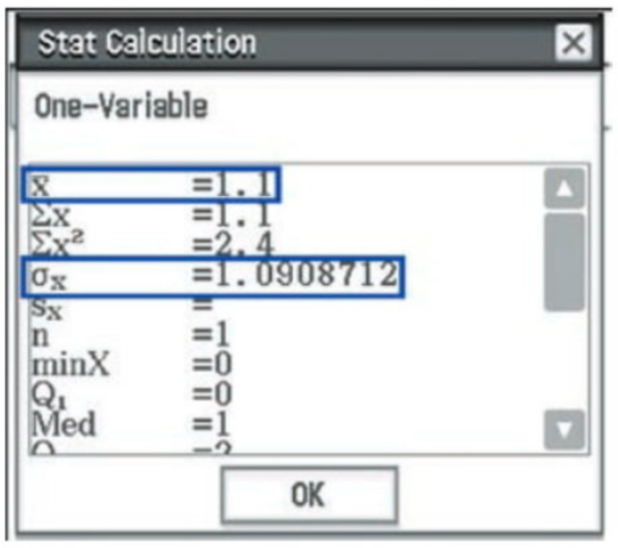
XList: list1

Freq: list2

4 Tap **Calc** > **One-Variable**.

5 Keep the **XList:** field set to **list1**.

6 Change the **Freq:** field to **list2**.



One-Variable

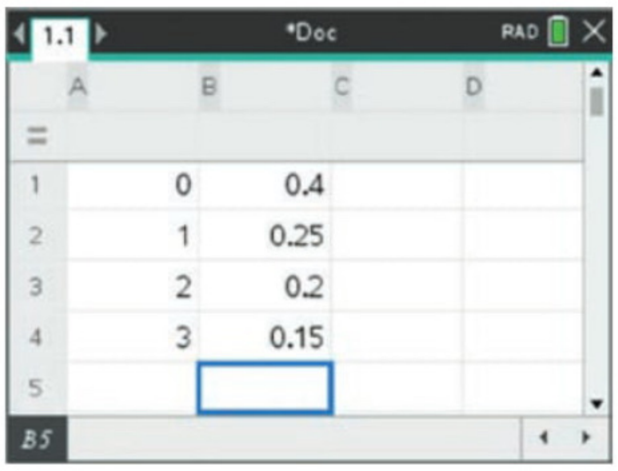
$\bar{x}$ =1.1
Σx =1.1
Σx^2 =2.4
σ_x =1.0908712
s_x =
n =1
minX =0
Q_1 =0
Med =1

7 The labels and corresponding values will be displayed.

8 The **mean** and **standard deviation** values are highlighted in blue.

9 To calculate the variance, square the standard deviation.

TI-Nspire

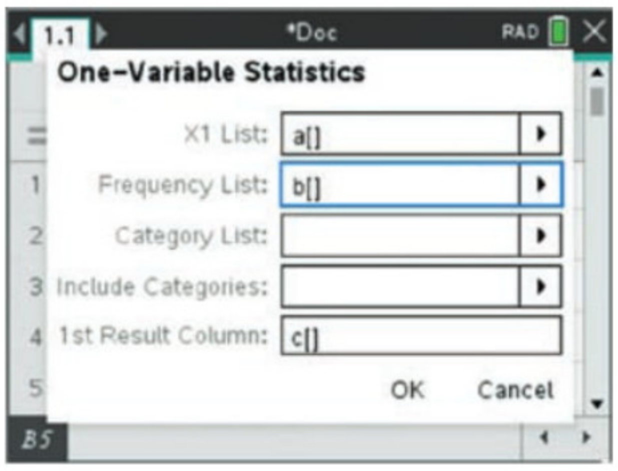


	A	B	C	D
=				
1	0	0.4		
2	1	0.25		
3	2	0.2		
4	3	0.15		
5				

1 Add a **List & Spreadsheet** page.

2 Enter the data from the table in columns **A** and **B** as shown above.

3 Press **menu** > **Statistics** > **Stat Calculations** > **One-Variable Statistics**.



One-Variable Statistics

X1 List: a[]

Frequency List: b[]

Category List:

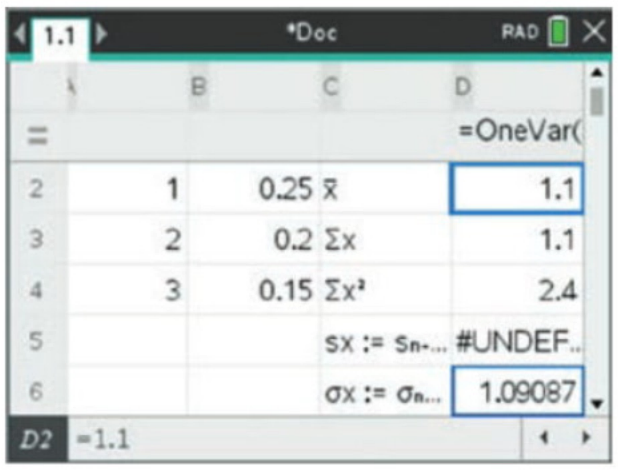
Include Categories:

1st Result Column: c[]

4 On the next screen, keep the **Num of Lists**: default setting of 1.

5 On the next screen shown above, in the **X1 List**: field, enter **a[ ]**.

6 In the **Frequency List**: field, enter **b[ ]**.



	A	B	C	D
=				=OneVar(
2	1	0.25	$\bar{x}$	1.1
3	2	0.2	Σx	1.1
4	3	0.15	Σx^2	2.4
5			sx := sn-...	#UNDEF...
6			σx := σn...	1.09087

7 The labels and corresponding values will be displayed in columns **C** and **D**.

8 The **mean** and **standard deviation** values are highlighted in blue.

9 To calculate the variance, square the standard deviation.

mean = 1.1, variance = 1.19, standard deviation = 1.09

EXERCISE 5.3 Measures of centre and spread

ANSWERS p. 397

Recap

1 On any given hour of the day, the number, X, of text messages that Danni receives is a random variable with probability distribution given by

x	0	1	2	3
$P(X = x)$	0.1	0.3	0.4	0.2

The probability that Danni receives five text messages over two consecutive hours is

A 0.08 **B** 0.09 **C** 0.16 **D** 0.3 **E** 0.6

2 The discrete random variable X has a probability distribution given by

x	1	2	3	4
$P(X = x)$	0.15	$4p$	0.25	$2p$

The value of p is

A $\frac{1}{15}$ **B** 0.1 **C** $\frac{1}{6}$ **D** 0.4 **E** 0.6

Mastery

3 WORKED EXAMPLE 10 Find the expected value of the discrete random variable X with the following probability distribution.

x	0	1	2	3	4	5
$P(X = x)$	0.02	0.13	0.38	0.17	0.05	0.25

4 WORKED EXAMPLE 11 The probability distribution of a discrete random variable X is given by the table below. Find the values of a and b if the mean of the distribution is 6.4.

x	5	6	7	8	9
$P(X = x)$	a	0.35	b	0.15	0.1

5 WORKED EXAMPLE 12 For the given probability distribution of a discrete random variable X, find the

x	1	2	3	4
$p(x)$	0.1	0.4	0.1	0.4

a expected value

b variance

c standard deviation, correct to three decimal places.

6 WORKED EXAMPLE 13 Find the variance of the discrete random variable X.

x	0	1	2	3
$p(x)$	0.1	0.2	0.3	0.4

▶

7 WORKED EXAMPLE 14 At the Mt Magnet archery competition, competitors are scored for accuracy and technique. The scores at this competition are a discrete random variable X which has an expected value of 15 and a variance of 12. Find

a $E(2X - 3)$

b $\text{Var}(2X - 3)$

The scores need to be rescaled to be compatible with other archery competitions in Western Australia. The scaling $Y = aX + b$, where a and b are positive constants, is applied so that the mean is 100 and the variance is 48.

c Find the values of a and b.

8 Using CAS 2 The discrete random variable X has a probability distribution given by the following table.

x	5	10	15	20
$P(X = x)$	0.12	0.25	0.28	0.35

Find the mean, the variance and the standard deviation of X, correct to two decimal places.

Calculator-free

9 (2 marks) On any given day, the number X of telephone calls that Daniel receives is a random variable with probability distribution given by

x	0	1	2	3
$P(X = x)$	0.2	0.2	0.5	0.1

Find the mean of X.

10 (3 marks) The probability distribution of a discrete random variable X is given by the table below.

x	0	1	2	3	4
$P(X = x)$	0.2	$0.6p^2$	0.1	$1 - p$	0.1

Let $p = \frac{2}{3}$.

a Calculate $E(X)$. (2 marks)

b Find $P(X \geq E(X))$. (1 mark)

11 © SCSA MM2018 Q4 (4 marks) Ten shop owners in a coastal resort were asked how many extra staff they intended to hire for the next holiday season. Their responses are shown below.

3, 0, 2, 1, 2, 1, 1, 0, 2, 1

If N = number of additional staff,

a copy and complete the probability distribution of N below. (2 marks)

n	0	1	2	3
$P(N = n)$				

b what is the mean number of staff the shop owners intend to hire? (2 marks)

Calculator-assumed

12 © SCSA MM2016 Q17 (7 marks) A school has analysed the examination scores for all its Year 12 students taking Methods as a subject. Let $X =$ the examination percentage scores of all the Methods Year 12 students at the school. The school found that the mean was 75 with a standard deviation of 22.

Determine

a $E(X + 5)$ (1 mark)

b $\text{Var}(25 - 2X)$. (2 marks)

The school has decided to scale the results using the transformation $Y = aX + b$ where a and b are constants and $Y =$ the scaled percentage scores. The aim is to change the mean to 60 and the standard deviation to 15.

c Determine the values of a and b. (4 marks)

13 (10 marks) Victoria Jones runs a small business making and selling statues.

The statues are made in a mould, then finished (smoothed and then hand-painted using a special gold paint) by Victoria herself. Victoria sends the statues **in order of completion** to an inspector, who classifies them as either 'Superior' or 'Regular', depending on the quality of their finish.

If a statue is Superior, then the probability that the next statue completed is Superior is p.

If a statue is Regular, then the probability that the next statue completed is Superior is $p - 0.2$. On a particular day, Victoria knows that $p = 0.9$.

On that day

a if the **first statue inspected is Superior**, find the probability that the third statue is Regular. (2 marks)

b if the **first statue inspected is Superior**, find the probability that the next three statues are Superior. (1 mark)

On another day, Victoria finds that if the **first statue inspected is Superior** then the probability that the third statue is Superior is 0.7.

c **i** Show that the value of p on this day is 0.75. (3 marks)

On this day, a group of three consecutive statues is inspected. Victoria knows that the **first** statue of the three statues is **Regular**.

ii Find the expected number of these three statues that will be Superior. (4 marks)

5.4 The Bernoulli and binomial distributions

The Bernoulli distribution

The **Bernoulli distribution** is a discrete distribution that has two possible outcomes, $x = 1$ and $x = 0$. The outcome $x = 1$, described as success, has probability p and the outcome $x = 0$, described as failure, has probability $1 - p$. The notation for a Bernoulli distribution is $X \sim \text{Bern}(p)$.

An example of a Bernoulli distribution is the single toss of a coin where we observe whether the outcome of a head occurs. The discrete random variable X can be thought of as the number of heads obtained. We will either get 1 head or 0 heads. The outcome of a head is success $(x = 1)$ and the outcome of a tail is failure $(x = 0)$.

However, a Bernoulli random variable will not necessarily always be about the number of times something occurs. For example, a light switch being in the ON position could be considered a success $(x = 1)$ and the OFF position could be considered a failure $(x = 0)$.

A **Bernoulli trial** is a trial, or result, of a Bernoulli distribution.

The probability distribution function for a Bernoulli distribution is

$$P(X = x) = p^x(1 - p)^{1-x} \text{ where } x = 0, 1.$$

mean: $\mu = p$ variance: $\sigma^2 = p(1 - p)$

Video playlist The Bernoulli and binomial distributions

Worksheets The Bernoulli distribution

Binomial probability experiments

Using the binomial probability distribution

WORKED EXAMPLE 15 Applying the Bernoulli distribution

One disc is removed from a box that contains 1 green and 5 blue discs. The discrete random variable X is the number of blue discs selected.

a Copy and complete the probability distribution for X shown below.

x	0	1
$P(X = x)$		

b State the distribution of X.

c Determine the mean and standard deviation of the distribution.

Steps	Working
a The probability of selecting a blue disc $(x = 1)$ is $\frac{5}{6}$ and the probability of a green disc $(x = 0)$ is $\frac{1}{6}$.	<table><tr><td>x</td><td>0</td><td>1</td></tr><tr><td>$P(X = x)$</td><td>$\frac{1}{6}$</td><td>$\frac{5}{6}$</td></tr></table>
b There are two possible outcomes. Either $x = 1$ or $x = 0$ so this satisfies the conditions for a Bernoulli distribution.	This is a Bernoulli distribution.

c Calculate the mean and variance.

The mean and variance can also be found using the formula

$\mathrm{E}(X) = p = \frac{5}{6}$

$\mathrm{Var}\,(X) = p(1 - p) = \frac{1}{6} \times \frac{5}{6} = \frac{5}{36}$

x	$p(x)$	$x \times p(x)$	$x^2 \times p(x)$
0	$\frac{1}{6}$	0	0
1	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{5}{6}$
Total		$\frac{5}{6}$	$\frac{5}{6}$

$\mathrm{E}(X) = \frac{5}{6}$

$\mathrm{E}(X^2) = \frac{5}{6}$

$\mathrm{Var}\,(X) = \sigma^2 = \mathrm{E}(X^2) - \mu^2 = \frac{5}{6} - \left(\frac{5}{6}\right)^2 = \frac{5}{36}$

The **binomial distribution** is a special discrete probability distribution of the results of a series of Bernoulli trials. The Bernoulli distribution is a special case of the binomial distribution where the number of trials (n) is 1.

Properties of binomial distributions

- Every outcome has two possibilities, which are categorised as 'success' or 'failure'.
- There is a series of n independent trials.
- The probability of success, denoted as p, remains constant on each trial.
- Selections with replacement produce a binomial distribution.
- The probability of x successes is given by $\mathrm{P}(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}$.
- The notation for a discrete random variable X with a binomial distribution is $X \sim \mathrm{Bin}(n, p)$, where n is the number of trials and p is the probability of success.

Exam hack

Always state the values of the **parameters** n and p in any binomial distribution question. This information may be written in the form $X \sim \mathrm{Bin}(n, p)$.

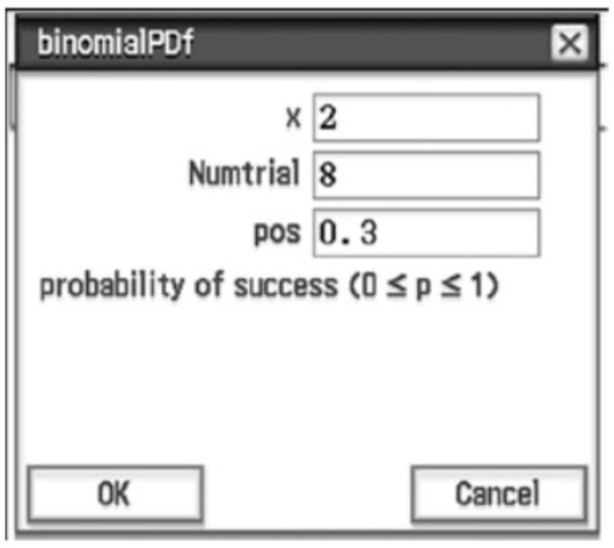
CAS can calculate probabilities for a binomial distribution.

USING CAS 3 The binomial distribution

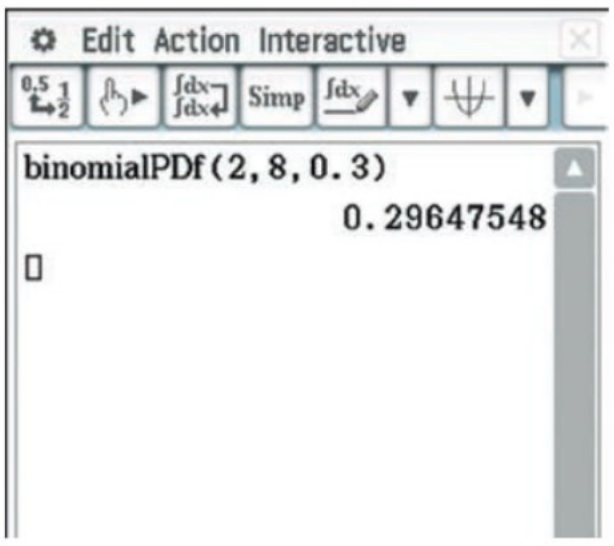
If $X \sim \text{Bin}(8, 0.3)$, find $P(X = 2)$ correct to three decimal places.

The notation $X \sim \text{Bin}(8, 0.3)$ means X has a binomial distribution, where $n = 8$ and $p = 0.3$.

ClassPad



x 2
Numtrial 8
pos 0.3
probability of success (0 ≤ p ≤ 1)



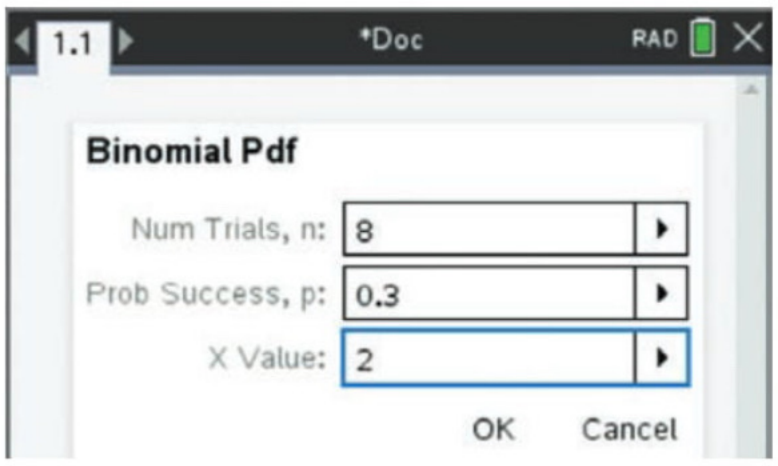
```
binomialPDf(2,8,0.3)
                    0.29647548
```

1 Tap **Interactive** > **Distribution/Inv.Dist** > **Discrete** > **binomialPDf**.

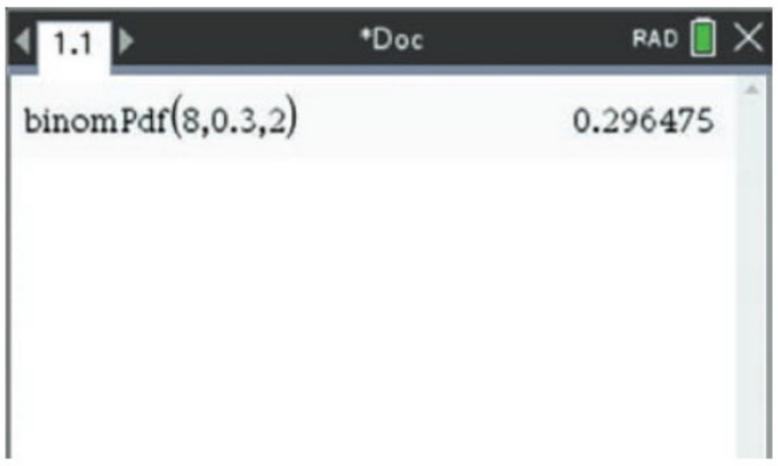
2 Enter the values as shown above.

3 The binomial probability will be displayed.

TI-Nspire



Binomial Pdf
Num Trials, n: 8
Prob Success, p: 0.3
X Value: 2



```
binomPdf(8,0.3,2)          0.296475
```

1 Press **menu** > **Probability** > **Distributions** > **Binomial Pdf**.

2 Enter the values as shown above.

3 The binomial probability will be displayed.

$P(X = 2)$ is 0.296 to three decimal places.

Probabilities for a binomial distribution using CAS

Probabilities	ClassPad	TI-Nspire
$P(X = x)$ Probability of single outcome	binomialPDf	Binomial Pdf
$P(X \le x)$ $P(X \ge x)$ $P(x_1 \le X \le x_2)$ Probability of a range of outcomes	binomialCDf	Binomial Cdf

WORKED EXAMPLE 16 Finding binomial probabilities

In a football match, a full forward has a probability of 0.35 of kicking a goal from a free kick.

If he is awarded 8 free kicks, in range of goal, in a match, find the probability, correct to three decimal places, that he kicks

a 2 goals

b at least 2 goals.

Steps | **Working**

a **1** The distribution is binomial because there are 2 outcomes on each trial: kick a goal or not kick a goal.

X = the number of goals scored from free kicks

$X \sim \text{Bin}(8, 0.35)$

Exam hack

Before solving the binomial distribution problem, write the values of n and p.

2 Write the probability using the formula:

$$\text{P}(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$\text{P}(X = 2) = \binom{8}{2}(0.35)^2(0.65)^6$$

3 Use CAS to calculate the answer.

Use the binomial probability density function binomialPDf (ClassPad) or Binomial Pdf (TI-Nspire), as the probability is a single outcome.

$\text{P}(X = 2) = 0.259$

b **1** Write the required probability.

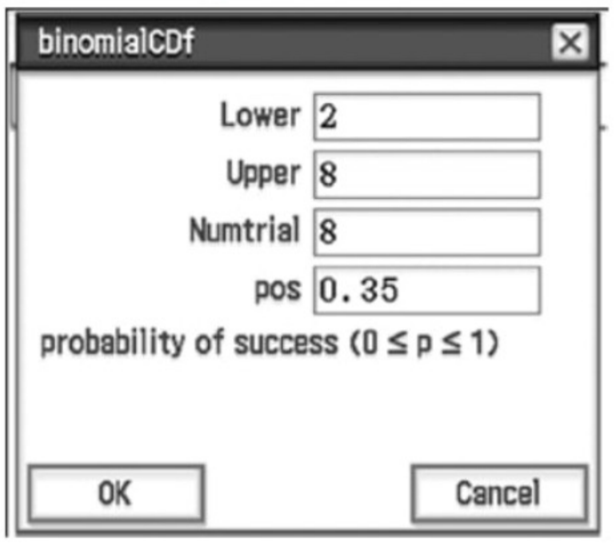
$\text{P}(X \geq 2)$

2 Use CAS to calculate the answer.

Use the binomial cumulative distribution function binomialCDf (ClassPad) or Binomial Cdf (TI-Nspire) as the probability is a range of outcomes.

The lower bound is 2 and the upper bound is 8. These bounds are inclusive.

ClassPad



binomialCDf

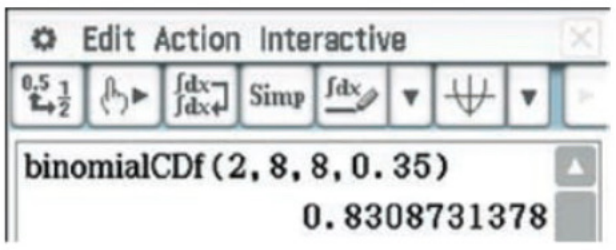
Lower: 2

Upper: 8

Numtrial: 8

pos: 0.35

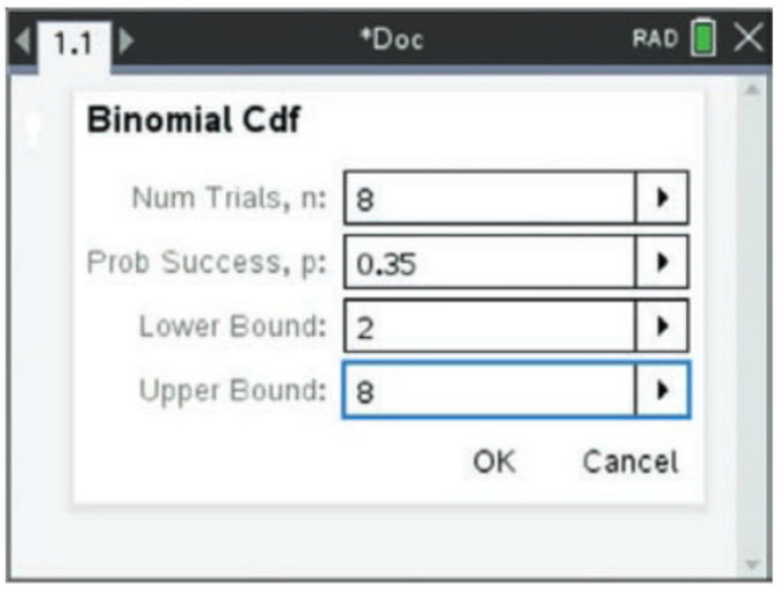
probability of success (0 ≤ p ≤ 1)



```
binomialCDf(2,8,8,0.35)
                0.8308731378
```

Tap **Interactive** > **Distribution/Inv.Dist** > **Discrete** > **binomialCDf** and enter the values shown.

TI-Nspire



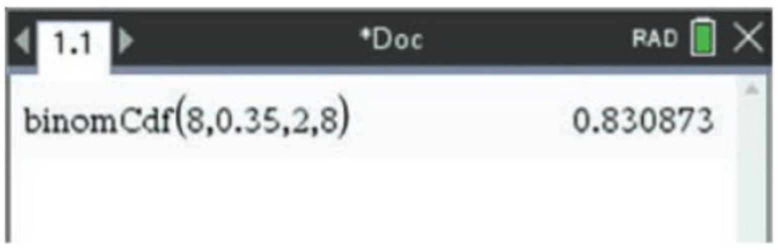
Binomial Cdf

Num Trials, n: 8

Prob Success, p: 0.35

Lower Bound: 2

Upper Bound: 8



```
binomCdf(8,0.35,2,8)        0.830873
```

Press **menu** > **Probability** > **Distributions** > **Binomial Cdf** and enter the values shown.

3 Write the probability correct to three decimal places.

$\text{P}(X \geq 2) = 0.831$

Technology-free binomial distribution problems

Binomial distribution probabilities can be calculated without CAS by using the binomial probability formula. This also requires a knowledge of **combinations,** which we learned in Year 11.

Combinations

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

To calculate, evaluate the fraction where the numerator is the product of r descending consecutive numbers starting with n and the denominator is r!

For example, $\binom{6}{3} = \frac{6!}{3! \times 3!} = \frac{6 \times 5 \times 4}{3 \times 2 \times 1}$.

WORKED EXAMPLE 17 Using the binomial distribution formula

A biased coin is tossed four times. The probability of a tail occurring on any toss is p.

a Find, in terms of p, the probability of obtaining
  i four tails
  ii three tails.

b Find the value of p if the probability of obtaining four tails is equal to the probability of obtaining three tails.

Steps	**Working**
a **i** **1** The distribution is binomial because there are two possible outcomes on each trial: a head or a tail.	X represents the number of tails. $X \sim \text{Bin}(4, p)$
2 Calculate the probability of $X = 4$ using the formula: $P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$	$P(X = 4) = \binom{4}{4}(p)^4(1-p)^0$ $= p^4$
ii Calculate the probability of $X = 3$ using the formula: $P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$	$P(X = 3) = \binom{4}{3}(p)^3(1-p)^1$ $= 4p^3(1-p)$ $= 4p^3 - 4p^4$
b Equate the answers in part **a** and solve for p.	$p^4 = 4p^3 - 4p^4$ $4p^3 - 5p^4 = 0$ $p^3(4 - 5p) = 0$ $p = 0, p = \frac{4}{5}$ $p = \frac{4}{5}$ (p cannot be zero)

Exam hack

Practise writing binomial probabilities using the formula, as this is often necessary when the given probability is a variable.

The mean and variance of a binomial distribution

Worksheets
Binomial probability – Mean and standard deviation

Applications of binomial probability

Mean and variance of a binomial distribution

If X is a discrete random variable with a binomial distribution, then

$$X \sim \text{Bin}(n, p)$$

mean: $\mu = np$

variance: $\sigma^2 = np(1-p)$

WORKED EXAMPLE 18 Finding the value of n and p for a binomial distribution

A binomial random variable has a mean of 12 and a variance of 9. Find the values of n and p.

Steps	Working
1 $\mu = np$ $\sigma^2 = np(1-p)$	$np = 12$ [1] $np(1-p) = 9$ [2]
2 Solve the equations by substitution to find the values of n and p.	Substitute [1] into [2]: $12(1-p) = 9$ $1 - p = \frac{9}{12}$ $p = 1 - \frac{9}{12}$ $= \frac{1}{4}$ Substitute into [1]: $n \times \frac{1}{4} = 12$ $n = 48$ So $n = 48$, $p = \frac{1}{4}$.

Finding the number of trials, n

Use CAS to find the number of trials for a binomial distribution.

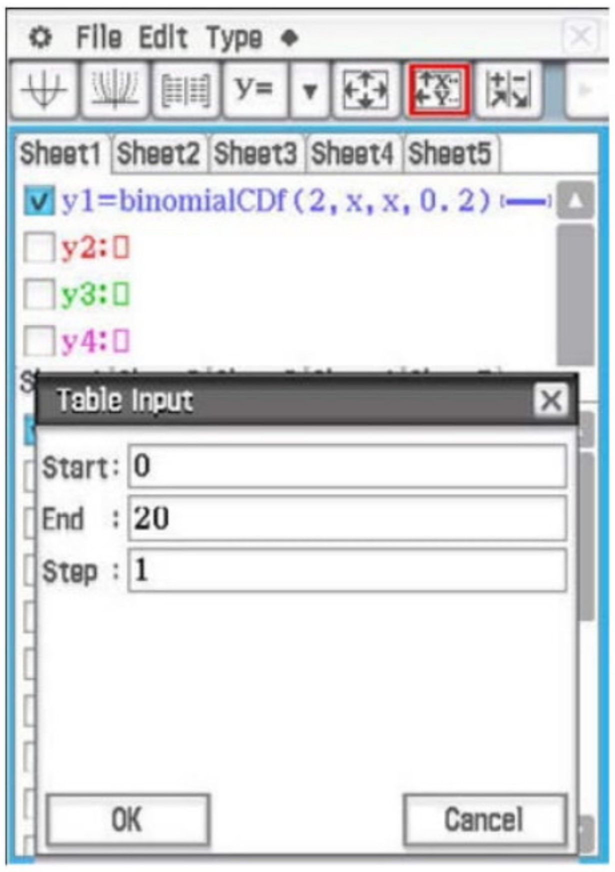
The problem is solved by creating a **discrete probability distribution** where the variable x represents the number of trials.

USING CAS 4 Finding the value of n for a binomial distribution

Amy plays basketball and practises shooting from the 3-point line. She finds during practice that her probability of making a 3-pointer is 0.2. Find the smallest number of shots Amy must take so that the probability she makes at least two 3-pointers is at least 0.72.

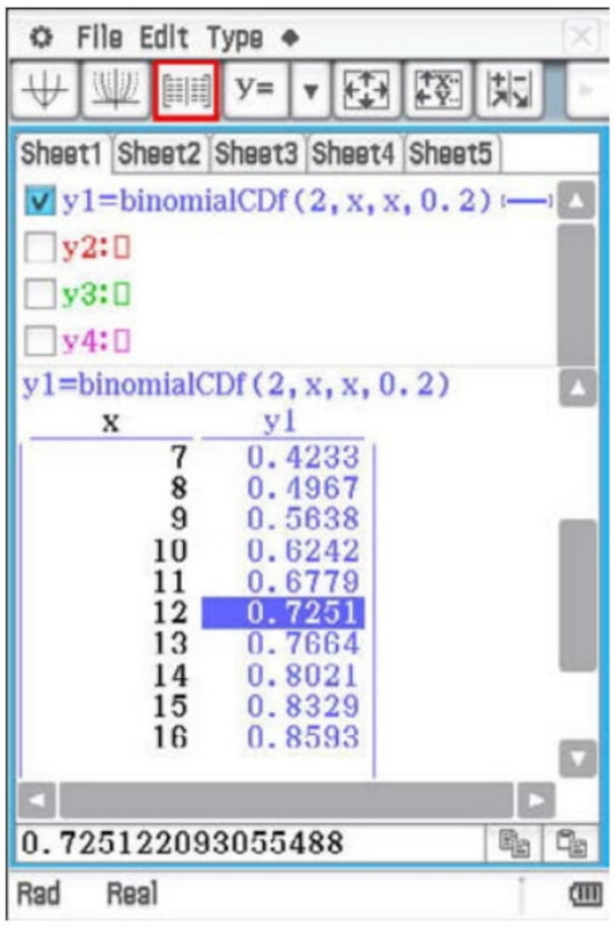
1 The problem is binomial because on each independent trial Amy will either hit or miss the 3-point shot. State the values of n and p and write the probability inequality.	Binomial, $n = x$, $p = 0.2$ X represents the number of 3-point shots scored. $P(X \geq 2) \geq 0.72$
2 The probability is a range of outcomes ($X \geq 2$), so use Binomial Cdf. $n = x$, $p = 0.2$, lower = 2, upper = x	$P(X \geq 2) = P(X = 2) + P(X = 3) + \ldots P(X = n)$ Using CAS notation $n = x$, $p = 0.2$, lower = 2 and upper = x. $\therefore$ Binomial Cdf $(x, 0.2, 2, x)$

ClassPad



☑ y1=binomialCDf(2, x, x, 0. 2)
☐ y2:□
☐ y3:□
☐ y4:□

Start: 0
End : 20
Step : 1



☑ y1=binomialCDf(2, x, x, 0. 2)
☐ y2:□
☐ y3:□
☐ y4:□

y1=binomialCDf(2, x, x, 0. 2)

x	y1
7	0. 4233
8	0. 4967
9	0. 5638
10	0. 6242
11	0. 6779
12	0. 7251
13	0. 7664
14	0. 8021
15	0. 8329
16	0. 8593

0. 725122093055488

1 Tap **Menu** > **Graph&Table**.

2 Open the **Keyboard** > **down arrow** > **Catalog**.

3 Tap **B** and scroll down to select **binomialCDf**.

4 Enter the values **(2,x,x,0.2)** as shown above.

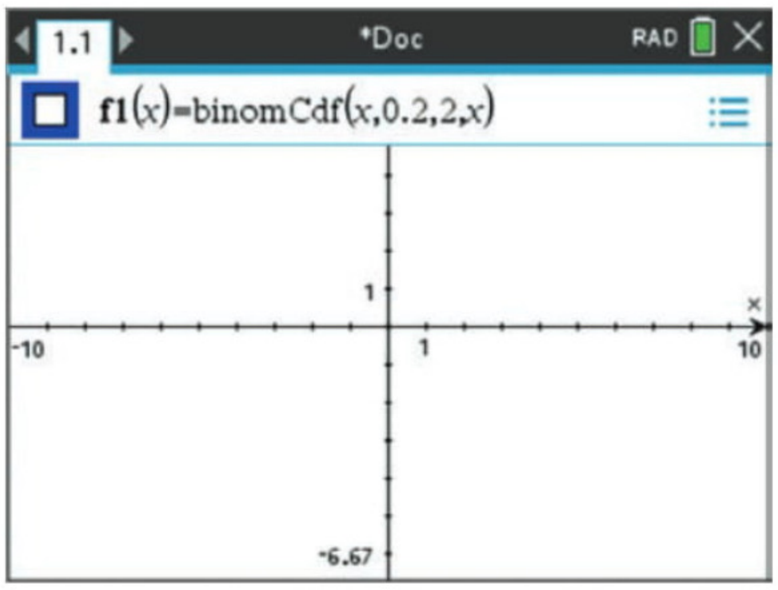
5 Tap **Table Input** to set a suitable range of values.

6 Tap **OK**.

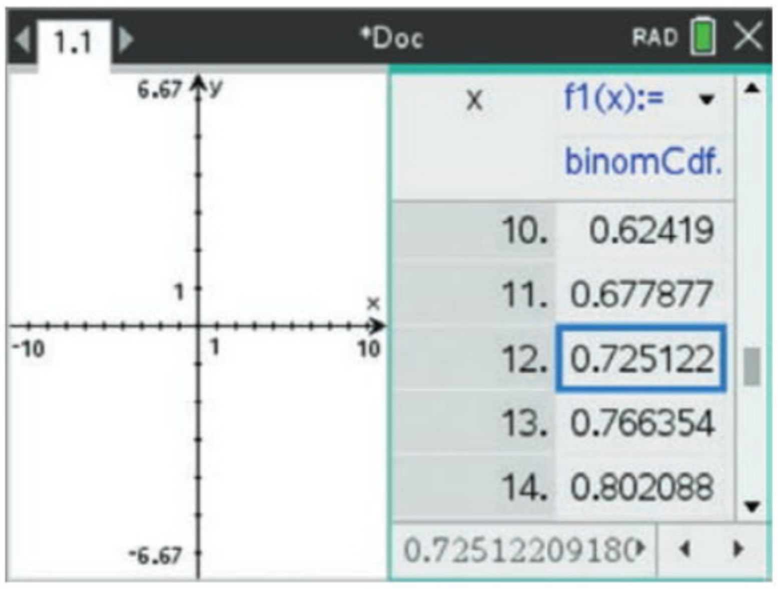
7 Tap **Table** to view the values.

8 Scroll down the table to the first binomialCDf value greater than 0.72.

TI-Nspire



f1(x)=binomCdf(x,0.2,2,x)



x	f1(x):= binomCdf.
10.	0.62419
11.	0.677877
12.	0.725122
13.	0.766354
14.	0.802088

0.7251220918O

1 Add a **Graphs** page.

2 Press **catalogue** then **B** to jump to the functions starting with b.

3 Scroll down and select **binomCdf**.

4 Enter the values as shown above.

5 After pressing **enter**, no graph will appear.

6 Press **menu > Table > Split-screen Table**.

7 Scroll down the table on the right to the first binomCdf value greater than 0.72.

The number of attempts required is 12.

Video WACE question analysis: Discrete random variables

WACE QUESTION ANALYSIS

© SCSA MM2019 Q10 Calculator-assumed (7 marks)

A group of researchers conducted a study into the number of siblings of adult Australian citizens. They surveyed a total of 200 participants and recorded the number of siblings, X, of each participant.

A few days later the lead researcher discovered that the survey data had been misplaced. Fortunately, one of the research assistants had been doing some rough calculations on a whiteboard and the lead researcher was able to recover the following information about the probability distribution for X and the mean μ.

x	0	1	2	3
$P(X = x)$	0.2	a	b	0.1

$$\mu = 1.3$$

The letters a and b have been used to denote unknown probabilities.

a **i** Write **two** independent equations for a and b. (2 marks)

ii Hence solve for the unknown probabilities. (2 marks)

Later that day the research assistant found the complete probability distribution in their records and discovered that they had made an error in their original calculation of the mean. The correct probability distribution is given in the table below.

x	0	1	2	3
$P(X = x)$	0.2	0.3	0.4	0.1

b **i** Given that there were 200 participants in the study, complete the table below to show the number of participants N with 0, 1, 2 and 3 siblings. (1 mark)

x	0	1	2	3
$P(X = x)$	0.2	0.3	0.4	0.1
N	40			

ii Determine the correct mean and standard deviation of the number of siblings X. (2 marks)

Reading the question

- Highlight the definition of both random variables. The random variable is discrete if there are a countable number of options.
- Highlight the nature of the answer required in each part. This is particularly important in parts where more than one answer is required.
- Notice, in part **b ii**, that the mean and standard deviation are for the variable X not N.

Thinking about the question

- You will need to know the properties of a discrete probability distribution and how to calculate the mean.
- You will also need to be able to solve simultaneous linear equations.
- The question requires you to find the standard deviation from a discrete probability distribution table.
- Many probability questions can be answered using CAS; however, the number of marks in each part of the question will give an indication of the amount of working that must be shown. In part **b ii** there are only 2 marks allocated for two answers, which is an indication that CAS can be used.

Worked solution (✓ = 1 mark)

a **i** Probabilities add to 1.

$\mathbf{0.2 + a + b + 0.1 = 1}$ ✓

$a + b = 0.7$ [1]

The mean is 1.3.

$\mathbf{0.2(0) + a(1) + b(2) + 0.1(3) = 1.3}$ ✓

$a + 2b = 1$ [2]

ii From the first equation

$a = 0.7 - b$

Substituting into the second equation

$(0.7 - b) + 2b = 1$

$b = 0.3$

$a = 0.4$

$\mathbf{b = 0.3}$ ✓

$\mathbf{a = 0.4}$ ✓

b **i**

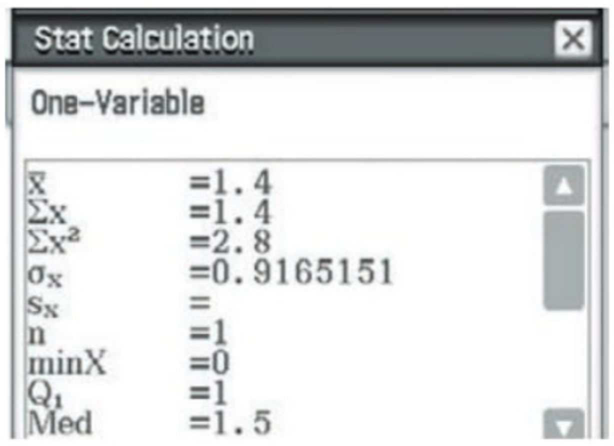
x	0	1	2	3
$P(X = x)$	0.2	0.3	0.4	0.1
N	40	**60**	**80**	**20**

determines all the correct N values ✓

ii $\mathbf{\mu = 1.4}$ ✓

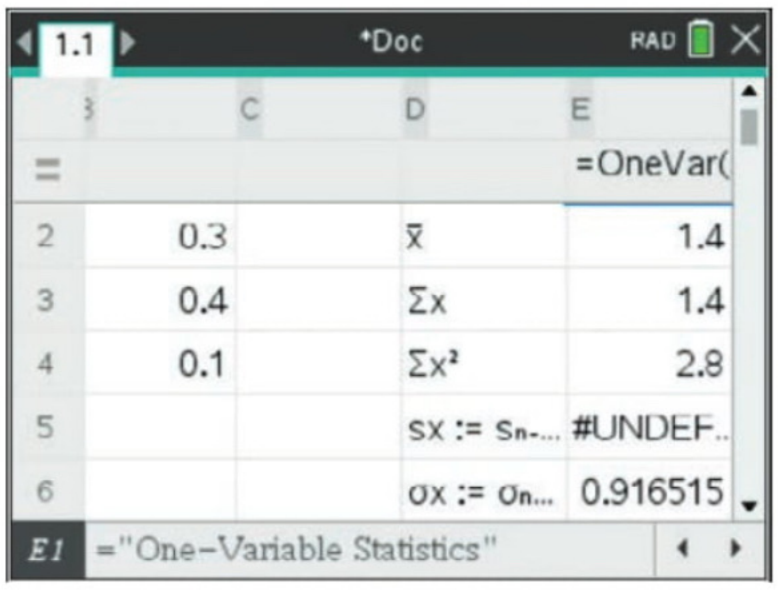
$\mathbf{\sigma = 0.9165}$ ✓

ClassPad



```
Stat Calculation
One-Variable
x̄      =1.4
Σx     =1.4
Σx²    =2.8
σx     =0.9165151
sx     =
n      =1
minX   =0
Q₁     =1
Med    =1.5
```

TI-Nspire



```
1.1   *Doc   RAD
   B     C    D              E
=                            =OneVar(
2  0.3        x̄              1.4
3  0.4        Σx             1.4
4  0.1        Σx²            2.8
5             sx := sn-...   #UNDEF..
6             σx := σn...    0.916515
E1 ="One-Variable Statistics"
```

EXERCISE 5.4 The Bernoulli and binomial distributions

ANSWERS p. 398

Recap

1 The discrete random variable X has the following probability distribution.

X	0	1	2	3	4	5	6
$P(X = x)$	0.05	0.13	0.27	0.1	0.25	0.14	0.06

The variance of X is

A 1.6091 **B** 2.5891 **C** 2.6125 **D** 3.03 **E** 11.77

2 The discrete random variable X has the following probability distribution, where $0 < p < \frac{1}{4}$.

X	0	1	2
$P(X = x)$	$3p$	p	$1 - 4p$

The mean of X is

A 1 **B** $2 - 3p$ **C** $1 - 4p$ **D** $4 - 15p$ **E** $2 - 7p$

Mastery

3 WORKED EXAMPLE 15 A building has one alarm and the probability that the alarm fails overnight is 0.05. Let the random variable X denote the number of times the alarm fails overnight.

a State the distribution of X.

b Find the mean and variance of X.

4 Using CAS 3

a If $X \sim \text{Bin}(15, 0.74)$, find $P(X = 12)$ correct to three decimal places.

b If $X \sim \text{Bin}(20, 0.625)$, find $P(X < 11)$ correct to three decimal places.

c If $Y \sim \text{Bin}(8, 0.4)$, find $P(1 < Y \leq 6)$ correct to three decimal places.

5 WORKED EXAMPLE 16 A test has eight multiple-choice questions with five possible answers each. A student guesses the answer to each question.

a Find the probability, correct to three decimal places, that the student guesses half the questions correctly.

A cyclist has a probability of 0.05 of puncturing a tyre on a training ride. During the month of April, the cyclist completes one training ride each day and the probability of puncturing a tyre on one day is independent of a puncture on any other day.

b Find the probability, correct to four decimal places, that the rider has at most two punctures during April.

6 WORKED EXAMPLE 17 Imogen is a soccer player who practises her penalty kicks many times each day. Each time she takes a penalty kick her probability of scoring a goal is p, independent of any other penalty kick. In one game Imogen had 6 penalty kicks.

a Find, in terms of p, the probability of Imogen scoring

i 5 goals **ii** 6 goals.

b Find the value of p if the probability of scoring 5 goals is equal to the probability of scoring 6 goals.

7 WORKED EXAMPLE 18

a A binomial random variable, X, has an expected value of 90 and a variance of 36. Find the parameters n and p.

b A binomial random variable, X, has an expected value of 10 and a variance of 9. Find $P(X \geq 15)$ correct to two decimal places.

8 Using CAS 4

a The probability of a darts player hitting a bullseye is $\frac{1}{8}$. What is the smallest number of darts the player should throw so that the probability of hitting at least one bullseye is greater than 0.8?

b A transport company claims that there is a 0.75 probability that each delivery it makes will arrive on time or earlier. Assume that whether each delivery is on time or earlier is independent of other deliveries. If the company makes n deliveries in a day, find the minimum value of n such that there is at least a 0.95 probability that one or more deliveries will **not** arrive on time or earlier.

Calculator-free

9 © SCSA MM2018 Q1 (9 marks) A bag contains one red marble and four green marbles. A single marble is drawn from the bag. The random variable Y is defined as the number of green marbles drawn from the bag.

a Copy and complete the probability distribution for Y shown below. (2 marks)

y	0	1
$P(Y = y)$		

b State the distribution of Y. (1 mark)

c Determine the mean and standard deviation of the distribution. (2 marks)

The above process is repeated five times, with the marble being replaced every time. The random variable X is defined as the number of green marbles drawn from the bag in five attempts.

d State the distribution of X, including its parameters. (2 marks)

e Evaluate the probability of selecting exactly two green marbles. (2 marks)

10 (3 marks) A paddock contains 10 tagged sheep and 20 untagged sheep. Four times each day, one sheep is selected at random from the paddock, placed in an observation area and studied, and then returned to the paddock.

a What is the probability that the number of tagged sheep selected on a given day is zero? (1 mark)

b What is the probability that at least one tagged sheep is selected on a given day? (1 mark)

c What is the probability that no tagged sheep are selected on each of six consecutive days?

Express your answer in the form $\left(\frac{a}{b}\right)^c$, where a, b and c are positive integers. (1 mark)

11 (4 marks) A biased coin is tossed three times. The probability of a head from a toss of this coin is p.

a Find, in terms of p, the probability of obtaining

  i three heads from the three tosses (1 mark)

  ii two heads and a tail from the three tosses. (1 mark)

b If the probability of obtaining three heads equals the probability of obtaining two heads and a tail, find p. (2 marks)

▶ **12** (2 marks) It is known that 50% of the customers who enter a restaurant order a cup of coffee. If four customers enter the restaurant, what is the probability that more than two of these customers order coffee? (Assume that what any customer orders is independent of what any other customer orders.)

13 (5 marks) Records of the arrival times of trains at a busy station have been kept for a long period. The random variable X represents the number of minutes **after** the scheduled time that a train arrives at this station; that is, the lateness of the train. Assume that the lateness of one train arriving at this station is independent of the lateness of any other train. The distribution of X is given in the table below.

x	-1	0	1	2
$P(X = x)$	0.1	0.4	0.3	p

a Find the value of p. (1 mark)

b Find $E(X)$. (1 mark)

c Find $Var(X)$. (1 mark)

d A passenger catches a train at this station on five separate occasions. What is the probability that the train arrives **before** the scheduled time on exactly four of these occasions? (2 marks)

Calculator-assumed

14 © SCSA MM2019 Q18 (9 marks) A building has five alarms configured in such a way that the system functions if at least two of the alarms work. The probability that an alarm fails overnight is 0.05. Let the random variable X denote the number of alarms that fail overnight.

a State the distribution of X. (2 marks)

b What assumptions are required for the distribution in part **a** to be valid? (2 marks)

c What is the probability that the alarm system fails overnight? (2 marks)

One of the alarms is removed in the evening for maintenance and is not replaced.

d What is the probability that the alarm system still works in the morning? (3 marks)

15 (5 marks) FullyFit is an international company that owns and operates many fitness centres (gyms) in several countries. At every one of FullyFit's gyms, each member agrees to have his or her fitness assessed every month by undertaking a set of exercises called **S**. There is a five-minute time limit on any attempt to complete **S** and if someone completes **S** in less than three minutes, they are considered fit.

At FullyFit's Ascot gym, it has been found that the probability that any member will complete **S** in less than three minutes is $\frac{5}{8}$. This is independent of any other member.

In a particular week, 20 members of this gym attempt **S**.

a Find the probability, correct to four decimal places, that at least 10 of these 20 members will complete **S** in less than three minutes. (2 marks)

b Given that at least 10 of these 20 members complete **S** in less than three minutes, what is the probability, correct to three decimal places, that more than 15 of them complete **S** in less than three minutes? (3 marks) ▶

▶ **16** (4 marks) A school has a class set of 22 new laptops kept in a recharging trolley. Provided each laptop is correctly plugged into the trolley after use, its battery recharges.

On a particular day, a class of 22 students uses the laptops. All laptop batteries are fully charged at the start of the lesson. Each student uses and returns exactly one laptop. The probability that a student does **not** correctly plug their laptop into the trolley at the end of the lesson is 10%. The correctness of any student's plugging-in is independent of any other student's correctness.

a Determine the probability that at least one of the laptops is **not** correctly plugged into the trolley at the end of the lesson. Give your answer correct to four decimal places. (2 marks)

b A teacher observes that at least one of the returned laptops is not correctly plugged into the trolley. Given this, find the probability that fewer than five laptops are **not** correctly plugged in. Give your answer correct to four decimal places. (2 marks)

17 (4 marks) Mani grows lemons, which are sold to a food factory. When a truckload of lemons arrives at the food factory, the manager randomly selects and weighs four lemons from the load. If one or more of these lemons is underweight, the load is rejected. Otherwise it is accepted.

It is known that 3% of Mani's lemons are underweight.

a Find the probability that a particular load of lemons will be rejected. Express the answer correct to four decimal places. (2 marks)

b Suppose that instead of selecting only four lemons, n lemons are selected at random from a particular load. Find the smallest integer value of n such that the probability of at least one lemon being underweight exceeds 0.5. (2 marks)

18 © SCSA MM2016 Q20ab (6 marks) A chocolate factory produces chocolates of which 80% are pink. Each box of chocolates contains exactly 30 pieces.

a Identify the probability distribution of X = the number of pink chocolates in a single box and also give the mean and standard deviation. (3 marks)

b Determine the probability, to three decimal places, that there are at least 27 pink chocolates in a randomly selected box. (3 marks)

19 (3 marks) Victoria Jones runs a small business making and selling statues.

The statues are made in a mould, then finished (smoothed and then hand-painted using a special gold paint) by Victoria herself. Victoria sends the statues **in order of completion** to an inspector, who classifies them as either 'Superior' or 'Regular', depending on the quality of their finish.

Victoria hears that another company, Shoddy Ltd, is producing similar statues (also classified as Superior or Regular), but its statues are entirely made by machines, on a construction line. The quality of any one of Shoddy's statues is independent of the quality of any of the others on its construction line. The probability that any one of Shoddy's statues is Regular is 0.8.

Shoddy Ltd wants to ensure that the probability that it produces at least two Superior statues in a day's production run is at least 0.9.

Calculate the minimum number of statues that Shoddy would need to produce in a day to achieve this aim. ▶

5 Chapter summary

Probability of an event

$$P(A) = \frac{\text{number of ways } A \text{ can occur}}{\text{total number of possible outcomes}}$$

Tree diagrams

A tree diagram is a useful way of representing a series of simple events. Each simple event is represented by another stage of branches with the number of branches in each set equal to the number of possible outcomes for the event.

Selections without replacement

Selection without replacement occurs when an item is selected from a sample and is not replaced, then another item is selected. The events are dependent.

The addition rule for probability

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Conditional probability formula

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

If events A and B are independent then $P(A|B) = P(A)$.

The properties of a probability distribution

- All probabilities must be between 0 and 1 inclusive: $0 \le p(x) \le 1$.
- The sum of all probabilities must equal 1: $\Sigma p(x) = 1$.

The expected value (mean)

- The **expected value** of X is written in mathematical notation as $E(X)$ or μ.

  $E(X) = \Sigma x \cdot p(x)$

The variance and standard deviation

- The computational formula for the **variance** of a discrete probability distribution:

  $\sigma^2 = \text{Var}(X) = E(X^2) - \mu^2 = \Sigma(x - \mu)^2 p(x)$

  where $E(X^2) = \Sigma x^2 \cdot p(x)$ and $\mu = E(X)$.
- The **standard deviation** $\sigma = \text{SD}(X) = \sqrt{\text{Var}(X)}$.

The expected value and variance of $aX + b$

- $E(aX + b) = aE(X) + b$
- $\text{Var}(aX + b) = a^2 \text{Var}(X)$

Bernoulli distribution

The Bernoulli distribution is a discrete distribution having two possible outcomes, $x = 1$ and $x = 0$. The outcome $x = 1$, described as success, has probability p and the outcome $x = 0$, described as failure, has probability $1 - p$. The notation for a Bernoulli distribution is $X \sim \text{Bern}(p)$.

If $X \sim \text{Bern}(p)$ then $E(X) = p$ and $\text{Var}(X) = p(1 - p)$.

The binomial distribution

The **binomial distribution** is a special discrete probability distribution of the results of a series of Bernoulli trials.

- Every outcome has two possibilities, which are categorised as 'success' or 'failure'.
- There is a series of n independent trials.
- The probability of success, denoted as p, remains constant on each trial.
- Selections with replacement produce a binomial distribution.
- The probability of x successes is given by

$$P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}$$

- The notation for a discrete random variable X with a binomial distribution is $X \sim \text{Bin}(n, p)$, where n is the number of trials and p is the probability of success.

Probabilities	ClassPad	TI-Nspire
$P(X = x)$ Probability of single outcome	binomialPDf	Binomial Pdf
$P(X \leq x)$ $P(X \geq x)$ $P(x_1 \leq X \leq x_2)$ Probability of a range of outcomes	binomialCDf	Binomial Cdf

The mean and variance of a binomial distribution

If X is a discrete random variable with a binomial distribution, then

mean: $\mu = np$

variance: $\sigma^2 = np(1 - p)$

Cumulative examination: Calculator-free

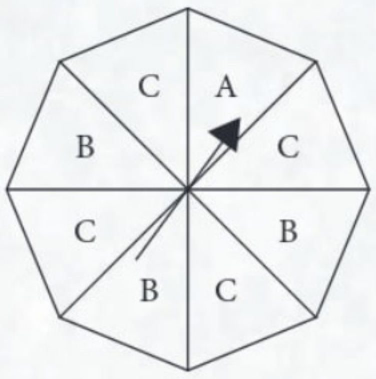
Total number of marks: 18 Reading time: 2 minutes Working time: 18 minutes

1 (2 marks) If $f(x) = (x^2 + 3)^2$, find $\frac{d^2y}{dx^2}$.

2 (3 marks) Let P be a point on the straight line $y = 2x - 4$ such that the length of OP, the line segment from the origin O to P, is a minimum. Find the coordinates of P.

3 © SCSA MM2020 Q1 (6 marks) Ashley and Xavier are playing a board game that requires them to use the spinner shown.

The player spins the arrowhead and the result is where the arrowhead is pointing when it stops moving. The diagram is showing a result of A.



a If the spinner is spun three times, what is the probability that B is never a result? (1 mark)

Let the random variable X be defined as the number of times B is the result when the spinner is spun three times.

b Copy and complete the table below showing the probability distribution of X. (3 marks)

x	0	1	2	3
$P(X = x)$				

c Determine the mean and variance of the distribution. (2 marks)

4 (3 marks) A car manufacturer is reviewing the performance of its car model X. It is known that at any given six-month service, the probability of model X requiring an oil change is $\frac{17}{20}$, the probability of model X requiring an air filter change is $\frac{3}{20}$ and the probability of model X requiring both is $\frac{1}{20}$.

a State the probability that at any given six-month service model X will require an air filter change without an oil change. (1 mark)

b The car manufacturer is developing a new model, Y. The production goals are that the probability of model Y requiring an oil change at any given six-month service will be $\frac{m}{m+n}$, the probability of model Y requiring an air filter change will be $\frac{n}{m+n}$ and the probability of model Y requiring both will be $\frac{1}{m+n}$, where m and n are positive integers.

Determine m in terms of n if the probability of model Y requiring an air filter change without an oil change at any given six-month service is 0.05. (2 marks)

5 (4 marks) For a certain population the probability of a person being born with the specific gene SPGE1 is $\frac{3}{5}$. The probability of a person having this gene is independent of any other person in the population having this gene.

a In a randomly selected group of four people, what is the probability that three or more people have the SPGE1 gene? (2 marks)

b In a randomly selected group of four people, what is the probability that exactly two people have the SPGE1 gene, given that at least one of those people has the SPGE1 gene?

Express your answer in the form $\frac{a^3}{b^4 - c^4}$, where a, b and c are positive integers. (2 marks)

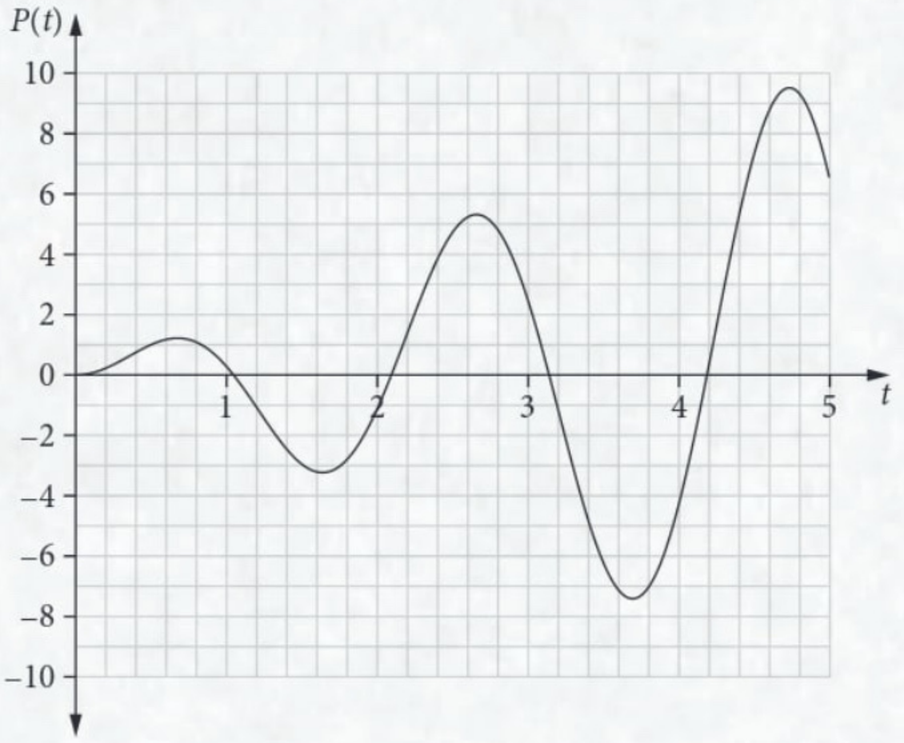
Cumulative examination: Calculator-assumed

Total number of marks: 30 Reading time: 3 minutes Working time: 30 minutes

1 © SCSA MM2019 Q7 (9 marks) A company's profit, in millions of dollars, over a five-year period can be modelled by the function:

$P(t) = 2t\sin(3t) \quad 0 \leq t \leq 5$ where t is measured in years.

The graph of $P(t)$ is shown below.



a Differentiate $P(t)$ to determine the marginal profit function, $P'(t)$. (2 marks)

b Calculate the rate of change of the marginal profit function when $t = \frac{\pi}{18}$ years. (4 marks)

c Use the increments formula at $t = \frac{7\pi}{6}$ to estimate the change in profit for a one month change in time. (3 marks)

2 © SCSA MM2017 Q13 (9 marks) Ravi runs a dice game in which a player throws two standard six-sided dice and the sum of the uppermost faces is calculated. If the sum is less than five, the player wins $20. If the sum is greater than eight, the player wins $10. Otherwise the player receives no money.

a Copy and complete the table below. (2 marks)

Amount won			
Probability			

b What is the expected amount of money won by a player each time they play? (2 marks)

c Liu Yang decides to play the game. If Ravi charges her $5 to roll two dice, who is likely to be better off in the long-term? Explain. (3 marks)

d If Ravi wants to make a long-term profit per game of 20% of what he charges, what should he charge a player to roll the two dice? (2 marks)

3 © SCSA MM2021 Q13abc (6 marks) A carnival game involves five buckets, each containing 5 blue balls and 15 red balls. A player blindly selects a ball from each bucket and wins the game if they select at least 4 blue balls. Let X denote the number of blue balls selected.

a State the distribution of X, including its parameters. (2 marks)

b What is the probability of a player winning the game on any given attempt? (2 marks)

c Players are charged \$2 for each attempt at the game and offered a \$150 prize if they win the game. By providing appropriate numerical justification, explain why this is not a good idea for the carnival organisers. (2 marks)

4 (6 marks) Doctors are studying the resting heart rate of adults in two neighbouring towns: Mathsland and Statsville. Resting heart rate is measured in beats per minute (bpm).

The doctors consider a person to have a slow heart rate if the person's resting heart rate is less than 60 bpm. The probability that a randomly chosen Mathsland adult has a slow heart rate is 0.1587. It is known that 29% of Mathsland adults play sport regularly. It is also known that 9% of Mathsland adults play sport regularly and have a slow heart rate. Let S be the event that a randomly selected Mathsland adult plays sport regularly and let H be the event that a randomly selected Mathsland adult has a slow heart rate.

a **i** Find $\mathrm{P}(H|S)$, correct to three decimal places. (1 mark)

ii Are the events H and S independent? Justify your answer. (1 mark)

b Find the probability that a random sample of 16 Mathsland adults will contain exactly one person with a slow heart rate. Give your answer correct to three decimal places. (2 marks)

Every year at Mathsland Secondary College, students hike to the top of a hill that rises behind the school.

Students who take less than 15 minutes to get to the top of the hill are categorised as 'elite'.

The probability of a student at Mathland Secondary College being categorised as 'elite' is 0.0266.

The Year 12 students at Mathsland Secondary College make up $\frac{1}{7}$ of the total number of students at the school. Of the Year 12 students at Mathsland Secondary College, 5% are categorised as elite.

c Find the probability that a randomly selected non-Year 12 student at Mathsland Secondary College is categorised as elite. Give your answer correct to four decimal places. (2 marks)